This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

TRIGONOMETRY

BÔCHER AND GAYLORD

TA
151
.L63

HARVARD UNIVERSITY

Library

OF

The School of Landscape Architecture

Recommended for Use with this Book

E. V. HUNTINGTON'S FOUR PLACE TABLES

OF LOGARITHMS AND TRIGONOMETRIC FUNCTIONS

IN WHICH ANGLES ARE EXPRESSED IN DEGREES AND DECIMAL PARTS OF A DEGREE, INSTEAD OF IN DEGREES, MINUTES, AND SECONDS

WITH AUXILIARY TABLES (CHIEFLY TO THREE FIGURES) OF

SQUARES, SQUARE ROOTS, CUBES, CUBE ROOTS, RECIPROCALS, CIRCUMFERENCES AND AREAS OF CIRCLES, EXPONENTIALS, NATURAL LOGARITHMS, RADIANS, AND CONSTANTS

COMPILED BY

E. V. HUNTINGTON

ASSISTANT PROFESSOR OF MATHEMATICS IN HARVARD UNIVERSITY

NINTH EDITION

Marginal Index — All the values of any one function at one opening of the book — Large, clear type, on special, dull finish paper — Special tables for small angles — Special facilities for interpolation — Abridged edition (without the auxiliary tables) regularly supplied to candidates in admission examinations at Harvard and Princeton Universities.

FOR SALE BY

THE HARVARD COÖPERATIVE SOCIETY

CAMBRIDGE, MASSACHUSETTS

Price, unabridged, in flexible cloth . . 60 cents
Abridged edition, bound in paper . . 35 cents

TRIGONOMETRY

WITH THE

THEORY AND USE OF LOGARITHMS

BY

MAXIME BÔCHER

PROFESSOR OF MATHEMATICS IN HARVARD UNIVERSITY

AND

HARRY DAVIS GAYLORD

MATHEMATICAL MASTER IN BROWNE AND NICHOLS SCHOOL
CAMBRIDGE

NEW YORK
HENRY HOLT AND COMPANY

Oct. 27, 1920
SCHOOL OF
LANDSCAPE ARCHITECTURE
HARVARD UNIVERSITY

2-132

COPYRIGHT, 1914
BY
HENRY HOLT AND COMPANY

2

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

Ref
QA
B23

PREFACE

TRIGONOMETRY, like all branches of mathematics, can be indefinitely expanded both by elaborating and multiplying theoretical details, and by extending the number and kind of applications. It does not follow that it is desirable that this be done. On the contrary, in view of our American conditions, where so much time has been lost by the boy before he begins the study of trigonometry, it is highly desirable that he master the essentials of the subject as rapidly as is consistent with thoroughness, in order that his progress to analytic geometry and the calculus be not unnecessarily delayed.

This textbook has been prepared with a view to giving an adequate treatment of what is essential, in a form sufficiently concise so that the real simplicity and brevity of the subject may be in evidence. The number of principles involved is very limited and should not be artificially multiplied by the formulation of rules.

The exercises, instead of being scattered through the text, are collected at the end of the book. They are divided into chapters and sections corresponding exactly to those of the text, so that when a certain section has been studied it is an easy matter to turn to the corresponding exercises. It is thought that the supply and variety of exercises will be found adequate for all ordinary purposes, and will allow the teacher a broad range of selection. We have kept in mind, however, that we were writing a textbook on trigonometry, and not on surveying, navigation, or astronomy.

A minimum course on logarithms and plane trigonometry may be obtained from Chapters I–IV by omitting the sec-

tions in small type. Some of the subjects which this course would omit, such as logarithms with other bases and circular measure, mean very little to a boy at this stage of his development and may well be postponed until he has begun the study of the calculus. We hope that teachers will regard it as a merit in our book that, starting with this minimum course, it enables them to go beyond it, so far as conditions may warrant, in almost any way which suits their purpose. Not only are the small type sections and the chapters on spherical trigonometry available for this, but an examination of the exercises will show numerous small matters which most teachers will rightly prefer to omit, but which may be used, especially with the better pupils, to extend their knowledge and stimulate their interest.

We have not striven for new methods, since those in common use are, in the main, good. At a few points, however, we think we have made improvements, as, for instance, in the proof of the formulas for the sine and cosine of $x+y$ where by using the conception of projections, *with which the student should have become thoroughly familiar at an earlier stage*, the proof (in particular the figure) becomes very simple while remaining sufficiently concrete.

We wish to thank Professors C. L. Bouton and E. V. Huntington of Harvard University and Mr. Wm. A. Francis of Phillips Exeter Academy who kindly read the manuscript and gave us the benefit of valuable suggestions and criticism.

CONTENTS

CHAPTER I

FUNCTIONS OF ACUTE ANGLES. TABLES OF NATURAL FUNCTIONS. SOLUTION OF RIGHT TRIANGLES

SECTION		PAGE
1.	Definitions	1
2.	Functions of the Complementary Angle	3
3.	Functions of 30°, 45°, and 60°	3
4.	Trigonometric Tables	5
5.	Four-place Tables	8
6.	Solution of Right Triangles	11
7.	Projections	13
8.	Line Values of the Functions	14
9.	Trigonometric Identities	16
10.	Trigonometric Equations	18

CHAPTER II

LOGARITHMS AND THEIR USE IN SOLVING RIGHT TRIANGLES

11.	Logarithms. Definitions and Fundamental Laws	20
12.	Logarithmic Tables	22
13.	Computation by Means of Logarithms	25
14.	Logarithms with Other Bases	28
15.	Solution of Right Triangles by Means of Logarithms	29

CHAPTER III

FUNCTIONS OF ANGLES OF ANY MAGNITUDE AND OF THE SUM OF TWO ANGLES

16.	Angles of Any Magnitude	32
17.	Positive and Negative Segments	33
18.	Functions of Angles of Any Magnitude	34
19.	Projections	36
20.	Line Values of Functions	38
21.	Reduction to Positive Acute Angles	40
22.	The Functions of $A \pm 90°$, $A \pm 180°$, etc.	41

SECTION PAGE
23. The Fundamental Identities 43
24. Circular Measure of Angles 44
25. Graphs of the Functions 45
26. Functions of the Sum and Difference of Two Angles . . . 46
27. Functions of $2x$ and $\frac{1}{2}x$ 49
28. Sums and Differences of Sines and Cosines 50
29. Trigonometric Identities 50
30. Trigonometric Equations 52
31. Anti-trigonometric Functions 54

CHAPTER IV

SOLUTION OF OBLIQUE TRIANGLES

32. The Law of Sines 56
33. The Law of Cosines 57
34. The Law of Tangents 58
35. Graphical Solution of Triangles 58
36. Case 1. Two Angles and One Side 61
37. Case II. Two Sides and the Angle Opposite One of Them . . 62
38. Case III. Two Sides and the Included Angle 65
39. Case IV. Three Sides 68
40. Area of a Triangle 72

CHAPTER V

SPHERICAL RIGHT TRIANGLES

41. Geometrical Introduction 73
42. Graphical Solution of Right Spherical Triangles 76
43. The Trigonometric Formulas 80
44. Spherical Right Triangles which are almost Plane 83
45. Napier's Rules, and the Trigonometric Solution of Spherical Right Triangles 84
46. Special Oblique Triangles Solved by Means of Right Triangles . 87

CHAPTER VI

THE OBLIQUE SPHERICAL TRIANGLE

47. The Law of Sines 88
48. The Law of Cosines 89
49. Formulas for the Half-angles 90

SECTION PAGE
50. The Solution of Spherical Triangles 91
51. Ambiguous Cases 93
52. The Law of Tangents 95
53. Dual Formulas 95
54. Napier's Analogies 96
55. Improved Solutions of Triangles 97

EXERCISES

Exercises on Chapter I 99
Exercises on Chapter II 107
Exercises on Chapter III 114
Exercises on Chapter IV 125
Exercises on Chapter V 134
Exercises on Chapter VI 138

TRIGONOMETRY

CHAPTER I

FUNCTIONS OF ACUTE ANGLES. TABLES OF NATURAL FUNCTIONS. SOLUTION OF RIGHT TRIANGLES

1. Definitions. Let A (Fig. 1) be any acute angle. If from points P_1, P_2, P_3 on one side of this angle perpendiculars be dropped on the other side, right triangles AP_1R_1, AP_2R_2, AP_3R_3 are formed which are all similar to each other. Hence the ratios of corresponding sides of these triangles are equal, as, for instance:

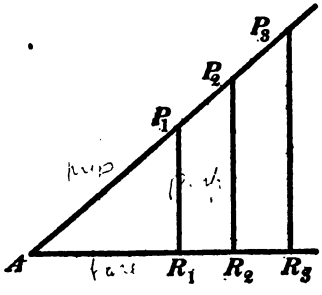


FIG. 1

$$(1)\qquad \frac{R_1P_1}{AP_1}=\frac{R_2P_2}{AP_2}=\frac{R_3P_3}{AP_3},$$

$$(2)\qquad \frac{AR_1}{AP_1}=\frac{AR_2}{AP_2}=\frac{AR_3}{AP_3},$$

$$(3)\qquad \frac{R_1P_1}{AR_1}=\frac{R_2P_2}{AR_2}=\frac{R_3P_3}{AR_3}.$$

The value of the ratios in (1) is called the Sine of A and denoted by $\sin A$. The value of the ratios in (2) is called the Cosine of A and denoted by $\cos A$. The value of the ratios in (3) is called the Tangent of A and denoted by $\tan A$. In order to obtain the values of these quantities it is clearly sufficient to form any one right triangle of which

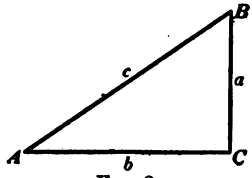


FIG. 2

A is an acute angle, for instance, ABC (Fig. 2). Such a triangle is called a triangle of reference for the given angle A. Our definitions may then be stated as follows:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}, \tag{4}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}, \tag{5}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}. \tag{6}$$

These are the three primary trigonometric functions. The three secondary trigonometric functions: Cotangent of A (ctn A or cot A), Secant of A (sec A), Cosecant of A (csc A), are defined as their reciprocals:

$$\csc A = \frac{1}{\sin A}, \tag{7}$$

$$\sec A = \frac{1}{\cos A}, \tag{8}$$

$$\operatorname{ctn} A = \frac{1}{\tan A}. \tag{9}$$

By comparing these formulas with the three preceding ones, we see that

$$\csc A = \frac{c}{a}, \tag{10}$$

$$\sec A = \frac{c}{b}, \tag{11}$$

$$\operatorname{ctn} A = \frac{b}{a}. \tag{12}$$

It is clear from these definitions that all the trigonometric functions of an acute angle are pure numbers independent of the unit of length used. We also notice that they are positive quantities, that the sine and cosine are always less than 1, and that the secant and cosecant are always greater than 1.

Very little used now are the functions known as the Versed sine and Coversed sine, which are defined by the equations

$$\text{vers } A = 1 - \cos A,$$
$$\text{covers } A = 1 - \sin A.$$

2. Functions of the Complementary Angle. If we apply the definitions of § 1 to the angle B instead of A, we find

$$\sin B = \frac{b}{c} = \cos A, \qquad \cos B = \frac{a}{c} = \sin A,$$
$$\tan B = \frac{b}{a} = \text{ctn } A, \qquad \text{ctn } B = \frac{a}{b} = \tan A,$$
$$\sec B = \frac{c}{a} = \csc A, \qquad \csc B = \frac{c}{b} = \sec A.$$

Thus the six trigonometric functions may be arranged in three pairs

sin, cos,
tan, ctn,
sec, csc,

the two functions of each pair being called co-functions of each other. If we recall that the angles A and B are complements (*i.e.* $A + B = 90°$), we may say that the trigonometric functions of the complementary angle are equal to the co-functions of the angle. Hence the name "cosine of A" as an abbreviation for "sine of the complement of A," and similarly for cotangent and cosecant.

3. Functions of 30°, 45°, and 60°. The functions of a few simple angles can be readily found.

Draw an isosceles triangle ABC in which $C = 90°$, $a = b = 1$. Then $A = B = 45°$, and $c = \sqrt{2}$. Hence the primary functions of 45° are

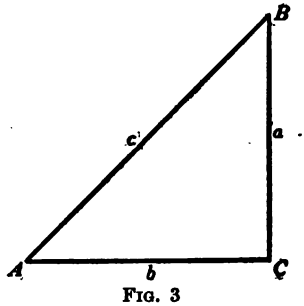


FIG. 3

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} = 0.707,$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} = 0.707,$$

$$\tan 45° = \frac{1}{1} = 1,$$

and the secondary functions are their reciprocals, $\csc 45° = \sqrt{2}$, $\sec 45° = \sqrt{2}$, $\operatorname{ctn} 45° = 1$.

Next draw an equilateral triangle ABD in which the sides have length 2. Bisecting this triangle by the altitude CB, we form a right triangle ABC whose angles A and B are 60° and 30°, respectively, while the sides are $AB = 2$, $AC = 1$, $CB = \sqrt{3}$. From this triangle we find the functions of 30° and 60°:

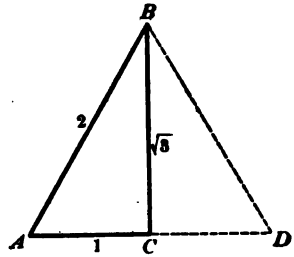


FIG. 4

$$\sin 30° = \cos 60° = \tfrac{1}{2} = 0.500,$$

$$\cos 30° = \sin 60° = \tfrac{1}{2}\sqrt{3} = 0.866,$$

$$\tan 30° = \operatorname{ctn} 60° = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} = 0.577,$$

$$\operatorname{ctn} 30° = \tan 60° = \sqrt{3} = 1.732,$$

$$\sec 30° = \csc 60° = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3} = 1.155,$$

$$\csc 30° = \sec 60° = 2 = 2.000.$$

We have now the data for a table of the functions of these three angles:

	sin	cos	tan
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$
45°	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$

The other functions, being the reciprocals of these, need not be tabulated if the values are expressed as radicals. If the decimal forms were used, however, it would be more difficult to write the reciprocals, and we should desire to have the other functions tabulated also.

4. Trigonometric Tables. For other angles, the method explained in the previous section is not applicable. If, for instance, we wish to find the functions of 10°, the only obvious way of proceeding is to draw with a protractor an angle of 10°, and to drop by means of suitable drawing instruments (either compasses and ruler or ruler and triangle) a perpendicular upon one side, from some point on the other side. If we measure accurately the lengths of the sides of the triangle thus formed, we can compute the functions of 10° with an accuracy depending on the degree of accuracy of our drawing. The student should construct in this way a table of one of the trigonometric functions for the angles 10°, 20°, 30°, . . . 80°.

The method just described is a very rough one. Other more refined methods, which we cannot describe here, have been invented, and by means of them tables computed, of which the following will serve as a sample:

	sin	tan	sec	
5°	0.0872	0.0875	1.00	85°
10°	0.174	0.176	1.02	80°
15°	0.259	0.268	1.04	75°
20°	0.342	0.364	1.06	70°
25°	0.423	0.466	1.10	65°
30°	0.500	0.577	1.15	60°
35°	0.574	0.700	1.22	55°
40°	0.643	0.839	1.31	50°
45°	0.707	1.00	1.41	45°
50°	0.766	1.19	1.56	40°
55°	0.819	1.43	1.74	35°
60°	0.866	1.73	2.00	30°
65°	0.906	2.14	2.37	25°
70°	0.940	2.75	2.92	20°
75°	0.966	3.73	3.86	15°
80°	0.985	5.67	5.76	10°
85°	0.996	11.4	11.5	5°
	cos	ctn	csc	

If we want to find from this table the functions of 15°, we look along the horizontal line, beginning on the extreme left with 15°, and read off directly sin 15° = 0.259, tan 15° = 0.268, sec 15° = 1.04. To find cos 15°, ctn 15°, and csc 15°, read the row opposite the 15° which occurs in the column at the extreme right, using the labels which occur at the foot of each column. Thus cos 15° = 0.966, ctn 15° = 3.73, csc 15° = 3.86.

Interpolation. If the given angle lies between the tabulated angles, the functions of this angle will lie between the values of the tabulated functions. For example, to find sin 17°, first look up sin 15° = 0.259 and sin 20° = 0.342, which are the nearest entries below and above 17°. The difference between 15° and 20° is 5°, and our angle, 17°, is $\frac{2}{5}$ of the way from 15° toward 20°. Hence the sine of 17° may

be expected to be approximately $\frac{2}{5}$ of the way from 0.259 toward 0.342. The difference between these two numbers is 0.083, and $\frac{2}{5}$ of this difference added to 0.259 gives us $0.259 + 0.033 = 0.292$, which is the sine of 17°. By a precisely similar process, the cosine of any angle between the tabulated values may be found. This process is called interpolation.

On account of the large and irregular differences between the consecutive tabulated values for tan, ctn, sec, and csc, interpolation for these functions is not accurate in this table, except between the first five or six entries.

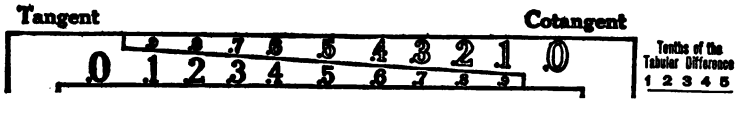
Inverse Use of the Table. When we are given some function of an unknown angle, the value of the angle may be found from the table as follows. Given $\sin x = 0.242$; to find the angle x. First note that there are in the sine column two numbers, $\sin 15° = 0.259$ and $\sin 10° = 0.174$, the first greater and the second less than 0.242. The difference between these numbers is 0.085, and the larger is greater than 0.242 by the amount 0.017. Hence 0.242 is $\frac{0.017}{0.085} = \frac{1}{5}$ of the way from 0.259 toward 0.174. Therefore the angle whose sine is 0.242 will be about $\frac{1}{5}$ of the way from 15° to 10°. Consequently $x = 14°$. In this way we may obtain the angle which corresponds to any given function, subject to the inaccuracy in the results of interpolation noted above for tan, ctn, sec, and csc.

Significant Figures. The table above is called a three-place table, or is said to be correct to three significant figures. The meaning of this expression can best be explained by reference to a specific example.

Take $\tan 60° = \sqrt{3} = 1.7321$, a number which cannot be accurately expressed in the decimal form. The first figure on the left which is not a zero is called the first significant figure. Hence 1 is the first significant figure in 1.7321. The position of the decimal point has no influence whatever on

the number of significant figures. Thus 7 is the second significant figure in this number, but is in the first decimal place. The first three significant figures will then be 1.73, and this is what we find for tan 60° in the above three-place table. If the fourth figure were 5, 6, 7, 8, or 9, the third significant figure would be increased by one unit if we wish to omit the subsequent figures.

5. Four-place Tables. A very convenient set of tables are Huntington's Four Place Tables.* The following is a section from the table of tangents and cotangents. It should be noticed that the angles are to be expressed not in degrees and minutes but in degrees and hundredths of a degree, the period in 42.16°, for instance, being a true decimal point. It is, of course, a simple matter of arithmetic to pass back and forth from this decimal system to the more common system of degrees and minutes, and even this work may be lessened by the use of a little auxiliary table printed at the beginning of Huntington's tables for easy conversion from one system to the other.



42	9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	9325	47	3 6 10 13 16
43	9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	0.9657	46	3 7 10 13 17
44	0.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	1.0000	45°	3 7 10 14 17
45°	1.0000												

Direct Use of the Tables. To find the value of tan 43.73° from this table, we look down the column of angles at the left just below the word tangent until we come to 43°. In this line, in the column headed .7 in black type at the top of the page, we find 0.9556 as the value of tan 43.70°. Our

* For sale by the Harvard Coöperative Society, Cambridge, Massachusetts. Price: Abridged edition, 35 cents, Complete edition, 60 cents.

angle, 43.73°, is three tenths of the way from 43.70° to 43.80°. Consequently we must add to 0.9556 three tenths of the difference 0.0034 between this entry and the next following one. That is

$$\tan 43.73° = 0.9556 + 0.0010 = 0.9566.$$

This process of interpolation can be somewhat shortened by using the little supplementary table in the right-hand margin. In the numerical example just mentioned we had to go $\frac{3}{10}$ of the distance from one tabular entry to the next. We look then in the column headed 3 in this marginal table, and find in it, against the line which we are using, the entry 10. This is the number we must add to the entry 0.9556 in our interpolation.

If we had been required to find tan 43.78°, instead of adding to tan 43.70° a correction corresponding to $\frac{8}{10}$ of the interval between the two tabular entries, it would be better to start from tan 43.80° = 0.9590 and subtract a correction corresponding to $\frac{2}{10}$ of the interval between the two tabular entries, and this correction, as we see from the marginal table, is 7 in the last place of decimals. Hence tan 43.78° = 0.9590 − 0.0007 = 0.9583.

To find the cotangent of an angle, we read the angle from the column of figures in outline type on the right directly under the word cotangent. In this case the angle increases as we read upward; and the tenths are given by the row of outline type across the top of the table, reading from right to left. Care must therefore be taken when interpolating between these tenths, because we are reading in directions opposite to those to which we are accustomed. For example, to find ctn 46.34°, we read ctn 46.30° = 0.9556 and ctn 46.40° = 0.9523. Hence, to find ctn 46.34°, we must subtract $\frac{4}{10}$ of the difference 0.0033, that is, 0.0013, from the number 0.9556, because we are going from right to left. We shall have then ctn 46.34° = 0.9543.

The tables for sine-cosine and secant-cosecant are used in a manner precisely similar to the one described above, and need no separate interpretation.

Inverse Use of the Tables. Suppose we have given $\tan x = 0.9875$, and we wish to find the number of degrees in the angle x. The nearest entries we find in the table are 0.9861 and 0.9896, the first of which corresponds to the angle 44.60°, the second to the angle 44.70°. The given value 0.9875 is 14 units in the last decimal place greater than the smaller entry, 0.9861. Looking in the same row on the extreme right in the column headed "Tenths of the Tabular Difference," we find this number 14, and since it is in the column headed 4, we see that the angle we are seeking is $\frac{4}{10}$ of the way from 44.60° to 44.70°. Hence

$$x = 44.64°.$$

If, instead, we had been given $\tan x = 0.9874$, the answer would have been the same; for the difference between this value and the nearest tabulated entry, 0.9861, is 13 in the last decimal place, and since 13 is not to be found among the "Tenths of the Tabular Difference" in this row, we must use the nearest value given, which is 14.

Again, suppose we had given $\operatorname{ctn} x = 0.9260$. This entry is found exactly in the table. Since we are using the table now as a table of cotangents, and not of tangents, we must obtain the whole number of degrees in x from the column just preceding the "Tenths of the Tabular Difference" on the right. The number of degrees in this case is therefore 47. To get the tenths of a degree, we follow up to the top of the page the column in which the entry 0.9260 is found, and read off the entry .2 which we there find in outline type. Our answer is thus $x = 47.20°$. If interpolation had been necessary, it would have been performed precisely as above.

In this table, and also in the table of secants and cosecants, the figure or figures to the left of the decimal point are not

tabulated except at infrequent intervals, except very near the end of the table. The reason is that this integral part of the function changes very slowly as we run down the table, and can thus be readily supplied from an occasional tabulation. When there is a change of one unit in this integral part, the fact is indicated by a special sign ⌜ inserted to draw attention to the change, as, for example, in the third row of the section reprinted above.

6. Solution of Right Triangles. By the *parts* of a triangle are meant its sides and angles. We shall regularly denote the sides of a triangle by the letters a, b, c and the angles by A, B, C; the angle A being always opposite a, B opposite b, and C opposite c. If, as is the case in this section, the triangle is given as a right triangle, we shall always suppose $C = 90°$.

To solve a triangle of which some parts are given, means to obtain the magnitudes of the other parts. The solution of triangles is one of the important applications of trigonometry, as indeed the word trigonometry — "the measurement of triangles" — indicates. For the moment it is only right triangles with which we shall be concerned. If, besides the right angle, any two parts, of which at least one is a side, are given, the triangle may readily be solved either graphically (by means of elementary geometrical constructions) or by means of trigonometry.

Graphical Solution. Given: $a = 24$, $b = 33$.
Find A, B, c.

By the use of triangle and ruler, or ruler and compasses, draw two straight lines perpendicular to each other at C (Fig. 5). From C measure off on one the length $CB = 24$, and on the other $CA = 33$. Draw BA. The line c can now be measured by means of a ruler, and the angles A and B by means of a protractor. These results will be more or

less accurate depending upon the degree of refinement in the instruments employed, and upon the care with which the work is done.

If any other two parts had been given, a similar method could be applied. For example,

Given: $A = 40°$, $c = 8$.
Find a, b, B.

With AB as one side and A as vertex, lay off the angle $BAC = 40°$ by means of a protractor. By means of a graduated ruler, lay off $AB = 8$. From B drop a perpendicular to AC meeting it at C. Then ABC is the desired right triangle. The other parts, a, b, and B, may be measured by means of a protractor or ruler.

Trigonometric Solution. Let ABC (Fig. 5) be a right triangle in which we know the sides $a = 24$ and $c = 41$, and the angle $C = 90°$. To find the length of the other side and the magnitudes of the other angles, write a function of the angle A which involves the known sides, a and c,

$$\sin A = \frac{a}{c} = \frac{24}{41} = 0.5854.$$

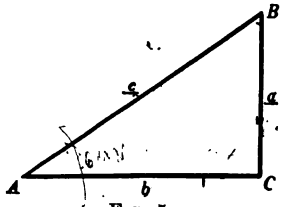


FIG. 5

Hence from the tables

$$A = 35.83°.$$

Then $$B = 90° - A = 54.17°.$$

To find b, we use the formula

$$\operatorname{ctn} A = \frac{b}{a}$$

Hence $$b = a \operatorname{ctn} A.$$

$$b = 24 \operatorname{ctn} 35.83° = 24 \times 1.3849 = 33.24.$$

The side b could also be found by the Pythagorean Theorem in the form $b^2 = c^2 - a^2$.

The method illustrated here applies with only slight variations to all cases. The only rule that needs to be remem-

bered is this: *Write down a function of one of the acute angles as a ratio of two sides, selecting the formula in such a way that of the three parts involved (the angle and the sides) two are known. Then solve the equation thus obtained for the unknown part.*

For instance, if A and b are given in Figure 5, we write

$$\cos A = \frac{b}{c}, \quad \text{or if we prefer} \quad \sec A = \frac{c}{b};$$

hence $c = \dfrac{b}{\cos A}$, or, from the second, $c = b \sec A$. These forms are equivalent to each other, but the second is preferable, since multiplication is easier to perform than division.

7. Projections. The projection of a point P on an indefinite line AB is the foot, M, of the perpendicular dropped from P on AB. If a point lies on AB, it must be regarded as its own projection. The projection of PQ on AB is the segment MN between the projections of P and Q. The length of this projection MN will obviously not be changed if the segment PQ is shifted in any way, without change of length, along the indefinite straight line on which it lies. Thus the projections of PQ, $P'Q'$, $P''Q''$ in Figure 6 are all of the same length. Let us call the positive acute angle between the two indefinite straight lines x. Then by shifting the segment PQ into the position $P''Q''$, where P'' is the point of intersection of the lines, we have in $P''Q''N''$ a triangle of reference for x, and hence

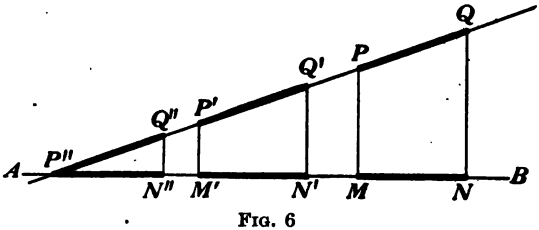


FIG. 6

$$\cos x = \frac{P''N''}{P''Q''} = \frac{MN}{PQ}.$$

That is

$$\overline{MN} = \overline{PQ} \cos x. \tag{1}$$

The length of the projection of a segment PQ of one line on another line AB is equal to the product of the length of PQ by the cosine of the angle between the lines.

8. Line Values of the Functions. In the ratios of § 1 which define the trigonometric functions, the numerators and denominators depend for their values on the unit of length used, but the ratios themselves are entirely independent of such a choice of a unit of length: they depend for their values solely on the magnitude of the angle considered. If, with the vertex of the angle as a center, and any selected unit as a radius, a circle is drawn, this circle will cut the side of the angle BAC (Fig. 7) at some point P_1. If now the perpendicular to the other side is dropped from P_1,

$$\sin A = \frac{R_1P_1}{AP_1},$$

and since AP_1 is the unit,

$$\sin A = \frac{R_1P_1}{1} = R_1P_1,$$

$$\cos A = \frac{AR_1}{1} = AR_1$$

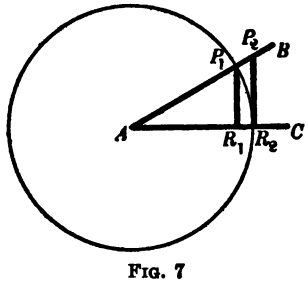


FIG. 7

and we have expressed the sine and cosine of A as the lengths of lines not as the ratios of two lengths.

To write $\tan A$ and $\sec A$ in a similar form, we must erect a perpendicular to the radius AR_2 at its extremity R_2 and let it cut AB at P_2. Then

$$\tan A = \frac{R_2P_2}{AR_2} = \frac{R_2P_2}{1} = R_2P_2,$$

$$\sec A = \frac{AP_2}{AR_2} = \frac{AP_2}{1} = AP_2.$$

The line values of csc A and ctn A are obtained by drawing (Fig. 8) AM perpendicular to AR_2 at A, and drawing a tangent to the circle at M meeting AB at P_3. Then since angle R_2AP_3 = angle MP_3A, being alternate interior angles of parallel lines,

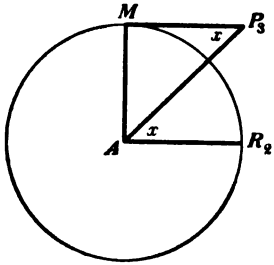


FIG. 8

$$\text{ctn } A = \frac{MP_3}{AM} = \frac{MP_3}{1} = MP_3,$$

$$\text{csc } A = \frac{AP_3}{AM} = \frac{AP_3}{1} = AP_3.$$

The Variation of the Trigonometric Functions. From the line values of the functions we can easily trace the change in values of the functions as the angle increases from 0° to 90°.

sin A, which is R_1P_1, begins at 0 for $A = 0°$ and increases to 1 for $A = 90°$.

cos A, or AR_1, begins at 1 for $A = 0°$ and decreases to 0 for $A = 90°$.

tan A, or R_2P_2, begins at 0 for $A = 0°$ and increases without limit as A approaches 90°.

sec A, or AP_2, begins at 1 for $A = 0°$ and increases without limit as A approaches 90°.

ctn A, or MP_3 (Fig. 8), is 0 for $A = 90°$ and increases without limit as A approaches 0°.

csc A, or AP_3, is 1 for $A = 90°$, and increases without limit as A approaches 0°.

These facts may be exhibited compactly in the following table :

	0°	90°
sin	0	1
cos	1	0
tan	0	∞

	0°	90°
csc	∞	1
sec	1	∞
ctn	∞	0

9. Trigonometric Identities. In any right triangle we have

$$(1) \qquad a^2 + b^2 = c^2,$$

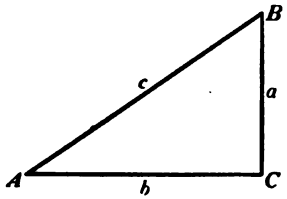


FIG. 9

or, dividing by c^2,

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

Since these two ratios are simply $\sin A$ and $\cos A$, respectively, this last equation may be written

$$(2) \qquad \sin^2 A + \cos^2 A = 1.$$

Similarly, dividing (1) by b^2 and using the definitions of $\tan A$ and $\sec A$, we find

$$(3) \qquad \tan^2 A + 1 = \sec^2 A.$$

Finally, dividing (1) by a^2 gives in the same way

$$(4) \qquad 1 + \operatorname{ctn}^2 A = \csc^2 A.$$

These three formulas are examples of what are known as trigonometric identities, since they are true for every acute angle A.

Another identity of equal importance may be obtained from the two formulas

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}$$

by dividing the first by the second. Using the definition of $\tan A$, we thus find

$$(5) \qquad \frac{\sin A}{\cos A} = \tan A.$$

Applications. By means of these four identities (2)–(5), together with the definitions, (7)–(9), § 1, of the secondary trigonometric functions, we can readily express any one of the trigonometric functions in terms of any other one. For

instance, if we wish to express $\cos A$ in terms of $\sin A$, we have merely to solve (2) for $\cos A$, getting

$$\cos A = \sqrt{1 - \sin^2 A},$$

the positive sign being used before the radical since the cosine of an acute angle is necessarily positive.

Similarly, to express $\tan A$ in terms of $\csc A$, we write

by (9), § 1 $$\tan A = \frac{1}{\operatorname{ctn} A},$$

by (4), § 9 $$\operatorname{ctn} A = \sqrt{\csc^2 A - 1}.$$

Hence by substitution

$$\tan A = \frac{1}{\sqrt{\csc^2 A - 1}}.$$

Further Identities. There are innumerable other trigonometric identities of greater or less importance which can be proved by means of the fundamental identities (2)–(5) and the definitions (7)–(9) of § 1. We give an illustration of the method of establishing the correctness of such identities:

To prove that

$$\tan^2 A - \sin^2 A = \tan^2 A \sin^2 A.$$

The left-hand side of this identity may be written by means of (5)

$$\frac{\sin^2 A}{\cos^2 A} - \sin^2 A = \sin^2 A\left[\frac{1}{\cos^2 A} - 1\right],$$

and this in turn by (8), § 1, becomes

$$\sin^2 A[\sec^2 A - 1].$$

On the other hand, the second member of the identity we wish to establish may, by (3), be written in precisely this same form. Thus the identity of the two sides is established.

Everywhere in mathematics where we wish to prove something, we must start from true premises (here formulas (2)–(5) and the defini-

tions of § 1) and deduce from them step by step the relation we wish to establish. Even if, by perfectly sound steps, we can deduce from this identity a true relation, this does not prove that the original identity was true, since a true result may perfectly well follow from a false premise by correct reasoning. Indeed, if we were willing to admit this kind of argument, everything, true or false, could be proved. For instance, to prove that $1 = 2$ multiply both sides by zero, getting $0 = 0$. This last is true, consequently, we should infer, the premise $1 = 2$ is true. Or we could proceed as follows: From the formula $1 = 2$ we obtain by subtracting $\frac{3}{2}$ from each side $-\frac{1}{2} = +\frac{1}{2}$. Squaring both sides gives $\frac{1}{4} = \frac{1}{4}$, a true relation. These illustrations ought to be enough to show how inconclusive is the method often used by inexperienced persons for proving a trigonometric identity, namely, to manipulate it by the aid of the relations (2)–(5) until it takes on a form which we recognize as correct.

10. Trigonometric Equations. The identities considered in the preceding section which are to be established as true for all acute angles A are very different in character from the equations we now wish to consider, which are true for only certain special angles which we may call their roots or solutions. Our problem in this section is to solve the trigonometric equations we consider, not to establish certain trigonometric identities.

The very simplest type is that in which the equation gives directly the numerical value of one of the trigonometric functions, as

$$\sin x = \tfrac{1}{2}, \quad \text{or} \quad \tan x = 2.1352.$$

These we can solve accurately (as in the first case just given where $x = 30°$) if the value happens to occur in the little table for 30°, 45°, 60° in § 3, or in the table for 0° or 90°, § 8; or approximately (as in the second case, where $x = 64.90°$) by means of the large tables.

A slightly more complicated case is illustrated by the equation

$$2 \sin^2 x - 3 \sin x + 1 = 0.$$

This may be regarded as a quadratic equation in $\sin x$.

Solving it by any of the ordinary methods of solving quadratic equations in algebra, we find

$$\sin x = 1 \text{ or } \tfrac{1}{2}.$$

The first of these values gives $x = 90°$, the second, $x = 30°$; and these are the two solutions of our trigonometric equation.

This method can always be used if only a single trigonometric function enters into our equation, and if the equation, when this function is taken as the unknown, is one we can solve by elementary algebra.

In most cases, however, several trigonometric functions enter into the given equation. Suppose, for instance, our equation is

$$2 \sin^2 x + 5 \cos x - 4 = 0.$$

By means of (2), § 9, we see that $\sin^2 x$ may be replaced by $1 - \cos^2 x$. The equation then becomes

$$2 \cos^2 x - 5 \cos x + 2 = 0.$$

Solving this as a quadratic equation in $\cos x$, we find

$$\cos x = 2 \text{ or } \tfrac{1}{2}.$$

Since, by § 1, the cosine of an acute angle cannot exceed 1, the equation $\cos x = 2$ has no solution. The only solution of our equation is therefore $x = 60°$, obtained from $\cos x = \frac{1}{2}$.

The general method here illustrated consists in throwing the equation, by means of the fundamental identities, into a form in which it involves only one trigonometric function. This would be done, to give one more illustration, in the case of the equation

$$2 \sin x = 5 \cos x$$

by dividing through by $\cos x$ and then using (5), § 9.

CHAPTER II

LOGARITHMS AND THEIR USE IN SOLVING RIGHT TRIANGLES

11. Logarithms. Definitions and Fundamental Laws. The solution of right triangles involves a considerable amount of numerical work if the numbers we have to use contain even as many as three or four significant figures, and we shall find that this is even more the case when we come to the methods of solving oblique triangles. One of the most efficient methods of shortening this and similar numerical work is the use of logarithms, invented for this purpose by John Napier, a Scotchman, in 1614.

If a is a positive quantity, the common or Briggs* logarithm of a is defined as that number x for which $a = 10^x$; or in words, it is the number which indicates *the power to which* 10 *must be raised in order to yield the number in question.* We write, then, $x = \log a$. Thus, for example, the logarithm of 100 is 2, because 10 has to be raised to the second power to give 100. Again $\log \frac{1}{10}$ is -1, since the (-1)th power of 10 is $\frac{1}{10}$.

Negative numbers have no logarithms.

The practical importance of logarithms lies in the fact that the relatively difficult numerical processes of multiplication, division, raising to a power, and extracting a root may be replaced respectively by the far simpler processes of addition, subtraction, multiplication, and division. This follows easily, as we shall soon see, from the following four fundamental properties of logarithms.

* Also called the *denary* logarithm.

(1) $\log ab = \log a + \log b.$

(2) $\log \frac{a}{b} = \log a - \log b.$

(3) $\log a^n = n \log a.$

(4) $\log \sqrt[n]{a} = \frac{1}{n} \log a.$

Proof of (1).

Let $\log a = x, \log b = y.$

Then, by the definition of a logarithm,

$$10^x = a, \qquad 10^y = b.$$

Hence $ab = (10^x)(10^y) = 10^{x+y}.$

Writing this in the logarithmic form, we have

$$\log ab = x + y,$$

and this is precisely $\log a + \log b$, as was to be proved.

Proof of (2).

Let $\log a = x, \log b = y.$

Then $10^x = a, \quad 10^y = b.$

Hence $\frac{a}{b} = \frac{10^x}{10^y} = 10^{x-y}.$

Consequently $\log \frac{a}{b} = x - y = \log a - \log b.$

Proof of (3).

Let $\log a = x.$

Then $10^x = a.$

Hence $a^n = (10^x)^n = 10^{nx}.$

Consequently $\log a^n = nx = n \log a.$

Proof of (4).

Since $\sqrt[n]{a} = a^{\frac{1}{n}}$, this formula follows at once from (3).

12. Logarithmic Tables. Before coming to the very important applications of these theorems, we must turn next to the practical question of how the logarithm of a number is actually to be found. By forming the positive and negative integral powers of 10, we get the first column of the following table, from which the second column follows by means of the definition of logarithms.

. . . . .	. . . . .
$10^4 = 10000$	$\log 10000 = 4$
$10^3 = 1000$	$\log 1000 = 3$
$10^2 = 100$	$\log 100 = 2$
$10^1 = 10$	$\log 10 = 1$
$10^0 = 1$	$\log 1 = 0$
$10^{-1} = 0.1$	$\log 0.1 = -1$
$10^{-2} = 0.01$	$\log 0.01 = -2$
$10^{-3} = 0.001$	$\log 0.001 = -3$
. . . . .	. . . . .

The second column may be regarded as a very rudimentary table of logarithms. The entries, however, are far too widely and irregularly spaced. Interpolation would be wholly impossible. Even from this table we may, nevertheless, infer certain important facts. In the first place we see that when $x > 1$, $\log x$ is positive; when $x < 1$ (x is always positive), $\log x$ is negative. As x increases indefinitely ($x = \infty$), $\log x$ also increases indefinitely, though very much more slowly. As x approaches zero, the numerical value of $\log x$ also increases indefinitely. These last two facts we may indicate briefly by writing

$$\log (+\infty) = +\infty, \quad \log 0 = -\infty.$$

We also see that the logarithm of any number between 1 and 10 is 0 plus a fraction; the logarithm of any number between 10 and 100 is 1 plus a fraction; etc. The only point remaining then is the determination of these fractions.

By methods which we cannot here explain, tables of the logarithms of numbers between 1 and 10 have been computed. These values are carried out to various numbers of decimal places according to the degree of accuracy we wish to secure. We will speak here merely of four-place tables, though the theory of five-place or larger tables is exactly the same.*

Direct Use of Four-place Tables. Suppose we want the logarithm of 3.263. We look down the column on the extreme left of page 16 until we find 3.2. On this line we run across to the column headed 6, and thus find

$$\log 3.26 = 0.5132.$$

Similarly $\log 3.27 = 0.5145.$

By interpolation between these two values ($\frac{3}{10}$ of the way from the first to the second) we find

$$\log 3.263 = 0.5136.$$

This interpolation may be performed either with or without the aid of the marginal table on the right headed "Tenths of the Tabular Difference."

We are thus in a position to use the table to find the logarithm of any number between 1 and 10. A little consideration will show that it may be used with almost the same ease to find the logarithm of any other number. For instance, if we want the logarithm of 326.3, we may write $326.3 = 100 \times 3.263$. Hence by Law (1) of § 11,

$$\log 326.3 = \log 100 + \log 3.263.$$

But $\log 100 = 2$, and, as we have just found from the table, $\log 3.263 = 0.5136$. Consequently

$$\log 326.3 = 2.5136.$$

The method here exemplified may be described by saying that, to find the logarithm of any number, we first shift the

* The details here refer to Huntington's tables, but all four-place tables of logarithms are pretty much alike.

decimal point until the number is reduced to a value between 1 and 10 and we compensate for this shifting by multiplying this last number by a suitable positive or negative integral power of 10. (Ex. $79320 = 7.932 \times 10^4$, $0.06943 = 6.943 \times 10^{-2}$.) The logarithm of the given number will then be the logarithm of the number between 1 and 10 plus the logarithm of the power of 10, which is an integer either positive or negative. Thus we see that the fractional part of the logarithm may be read off directly from the table by simply ignoring the decimal point both in the given number and in the table.*

Suppose then that we wish to find log 1450. We notice first that 1450 being between 1000 and 10000, the integral part of its logarithm is 3. The fractional part we read off directly from the table as 0.1614. Thus

$$\log 1450 = 3.1614.$$

Negative Characteristics. Suppose we want log 0.0145. This number is between 0.1 and 0.01. The logarithm will then be a little greater than $\log 0.01 = -2$. To this integral part, -2, we must therefore *add* the fractional part 0.1614, which we find from the table, so that

$$\log 0.0145 = -2 + 0.1614.$$

It must be carefully noticed that this is *not* the same as -2.1614. It may, for compactness, be written $\bar{2}.1614$, where the minus sign placed over the 2 applies to the integral part only.

Hence, first find the positive or negative integral part (characteristic) without reference to the table, by simply noting the position of the decimal point. Then find the positive fractional part (mantissa) by disregarding the decimal point and using the table.

Another form, which is often used in writing logarithms

* The decimal point in the table is thus, in practice, wholly useless. It is omitted in almost all tables except Huntington's.

with negative characteristics, consists in replacing the negative characteristic by a positive part to the left of the mantissa and a negative part to the right. Thus $\bar{2}.1614$ may be written $8.1614 - 10$, because $8 - 10 = -2$; or we may write $3.1614 - 5$, etc.

Inverse Use of the Table. Given $\log x = 2.3789$, to find x. Looking into the table we find 3784 in the fourteenth row from the top on page 16. This differs from our logarithm by 5, which corresponds to a difference of 3 in the last place of the corresponding number. The number corresponding to the logarithm, 3784, is 2390, hence the number we seek is $2390 + 3$, or $x = 239.3$. The decimal point is placed after the 9 because the characteristic, 2, of our logarithm (2.3789) indicates that the number x is between 100 and 1000.

13. Computation by Means of Logarithms. We give now some examples illustrating the way in which logarithms are to be used in performing certain numerical computations. These applications of logarithms depend directly, as will be seen, on formulas (1)–(4) of § 11.

Example 1. Compute the value of the product

$$2731 \times 0.05283.$$

From (1), § 11, we have

$$\log(2731 \times 0.05283) = \log 2731 + \log 0.05283$$
$$\log 2731 = 3.4364$$
$$\log 0.05283 = \bar{2}.7228$$
$$\log \text{ of product} = \overline{2.1592}.$$
$$\therefore \text{ product} = \boxed{144.3}$$

Three points may be noted in connection with this example. First, in adding the two logarithms, 3.4364 and $\bar{2}.7228$, we must remember that the characteristic (integral

part) of the second, but not the mantissa, is negative. Secondly, in finding the number whose logarithm is 2.1592, it is more convenient (though not necessary) to use, not the main table of logarithms on pages 16–17, but the supplementary table on pages 14–15 where the logarithms of numbers between 1 and 2 will be found tabulated at intervals of 0.001 instead of 0.01 as in the main table. The advantage of using this table whenever possible is that it makes interpolation unnecessary.

Finally we note that whenever a computation is performed by means of four-place logarithms, the answer should be given to four, and only four, significant figures.

Example 2. Compute the value of $x = \sqrt{\frac{(528)^3(231)}{1573}}$.

Here we first look up

$$\log 528 = 2.7226.$$

Multiplying this by 3, we have by (3), § 11,

$$\begin{aligned} \log (528)^3 &= \ 8.1678 \\ \log 231 &= \ 2.3636 \\ \log \text{ numerator} &= \overline{10.5314} \\ \log 1573 &= \ 3.1967 \\ \log \text{ fraction} &= \overline{\ 7.3347} \text{ by (2), § 11.} \end{aligned}$$

Now by (4), § 11,

$$\log x = \tfrac{1}{2} \log \text{ fraction} = 3.6674.$$

$$\therefore x = \boxed{4649}$$

Example 3. Compute the value of

$$x = \sqrt[3]{0.03824}.$$

$$\log 0.03824 = \bar{2}.5826.$$

This logarithm has a negative characteristic, -2, which is not divisible by 3. In such a case as this, it is best to use the form suggested near the end of § 12 in such a way as to

make the negative part exactly divisible by 3. Thus we may write

$$\log 0.03824 = 1.5826 - 3.$$

Now we have, by (4), § 11,

$$\log x = \tfrac{1}{3} \log 0.03824 = 0.5275 - 1.$$

$$\therefore x = \boxed{0.3369}$$

Cologarithms. It is often convenient, especially when several factors occur in the denominator of a fraction, to treat the fraction as a product, thus

$$\frac{(121.6)(9.025)}{(48.30)(3662)(0.08560)} = (121.6)(9.025)\left(\frac{1}{48.30}\right)\left(\frac{1}{3662}\right)\left(\frac{1}{.08560}\right).$$

To find the value of this fraction, we may proceed to add the logarithms of the five factors on the right. The logarithm of the third factor is obtained by subtracting log 48.30 from log 1, and similarly for the fourth and fifth factors. Since $\log 1 = 0$, we have

$$0.0000 - 1.6839 = \bar{2}.3161,$$

a subtraction which can easily be performed mentally. This little step has to be taken so often that it is worth while to get into the habit of performing it in the head as follows:

By the common process of *borrowing*.

	$0.0000 = \bar{1}.999(10)$
subtract	$1.683\ 9$
getting	$\bar{2}.316\ 1$

Since in this last subtraction no borrowing is necessary, we can perform it mentally *beginning at the left* and writing down the figures in the answer from left to right, remembering that the upper characteristic is always $\bar{1}$.

The number we thus obtain is called the cologarithm.

$$\operatorname{colog} 48.30 = \bar{2}.3161.$$

The cologarithm of any number is the logarithm of the reciprocal number. The work of finding the cologarithm should always be done in the head, it being almost as easy to write down the cologarithm from the table as the logarithm itself.

The computation of the example given above is as follows:

$$\begin{aligned}
\log 121.6 &= 2.0849 \\
\log 9.025 &= 0.9554 \\
\operatorname{colog} 48.30 &= \bar{2}.3161 \\
\operatorname{colog} 3662 &= \bar{4}.4363 \\
\operatorname{colog} 0.08560 &= \underline{1.0675} \\
\text{log of fraction} &= \bar{2}.8602 \\
\text{fraction} &= \boxed{0.07248}
\end{aligned}$$

Solution of Exponential Equations. So far we have used logarithms to shorten numerical computations which we already know how to perform by other methods. We come now to a question for which the only available method of solution is by logarithms.

By an exponential equation is meant an equation in which the unknown quantity appears in an exponent; for instance,

$$5^x = 12.$$

To solve this equation, take the logarithms of both sides, thus getting

$$x \log 5 = \log 12.$$

Hence
$$x = \frac{\log 12}{\log 5} = \frac{1.0792}{0.6990} = 1.544.$$

The last division here may be performed either by elementary methods or by means of logarithms.

14. Logarithms with Other Bases. The definition of logarithms given above and the theorems concerning them, but not the rules for the use of the tables, may be applied without change if, instead of using powers of 10 in our definition, we use powers of any positive quantity, c, (not equal to 1) which we then call the *base* of our system of logarithms. We may then say: The logarithm of any positive number N to the base c (written $\log_c N$) is the number x which indicates the power to which c must be raised to give N:

$$c^x = N.$$

It will be seen that if $c = 10$, we have the ordinary or denary logarithms defined in § 11. Formulas (1)–(4) of that section remain valid if the logarithms are taken not to the base 10, but to any base, c, the proofs requiring merely the substitution of c for 10.

It is not difficult by means of the ordinary tables (base 10) to find the value of a logarithm with any other base. Suppose, for instance, we want to find the value of

$$x = \log_2 7,$$

i.e. to find the value of x such that

$$2^x = 7.$$

This is an exponential equation of the form solved at the close of § 13. The method of solution there indicated is to take the (ordinary) logarithm of both sides

$$\log_{10} 2^x = \log_{10} 7,$$

or
$$x \log_{10} 2 = \log_{10} 7.$$

$\therefore$
$$x = \frac{\log_{10} 7}{\log_{10} 2} = \frac{0.8451}{0.3010} = \boxed{2.808}$$

The result here obtained for the bases 2 and 10 and the special number 7 may be obtained in exactly the same way in the general form

$$\log_b N = \frac{\log_a N}{\log_a b},$$

where a, b, N may be any positive numbers, provided neither a nor b are equal to 1.

In the higher parts of mathematics the base

$$e = \lim_{n = \infty} \left(1 + \frac{1}{n}\right)^n = 2.718+$$

is much used.

15. Solution of Right Triangles by Means of Logarithms. It is evident that the solution of triangles by this method will involve the logarithms of the functions of an angle. The tables are so arranged that we need not look up first the function and then its logarithm, but can look immediately into the tables labeled log sin, log cos, log tan, etc., and find the logarithm of the function required.

Example 1. Given $a = 36.00$,
$b = 27.60$.
Find A, B, c.

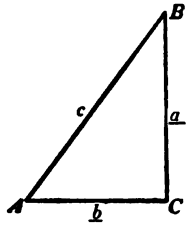


FIG. 10

$$\tan A = \frac{a}{b}$$
$$= \frac{36}{27.6}$$
$$\log 36 = 1.5563$$
$$\log 27.6 = 1.4409$$
$$\log \tan A = 0.1154$$
$$A = \boxed{52.52^\circ}$$
$$B = 90^\circ - A = \boxed{37.48^\circ}$$

$$\sin A = \frac{a}{c}$$
$$c = \frac{a}{\sin A}$$
$$= \frac{36}{\sin 52.52^\circ}$$
$$\log 36 = 1.5563$$
$$\log \sin 52.52^\circ = \bar{1}.8996$$
$$\log c = 1.6567$$
$$c = \boxed{45.36}$$

The hypotenuse, c, might also be obtained from the Pythagorean proposition, $c^2 = a^2 + b^2$. This formula, however, is not well adapted to logarithmic computation, and the trigonometric method just explained for getting c is usually better. An excellent check may be obtained by writing

$$b^2 = c^2 - a^2 = (c - a)(c + a).$$

In our case

$$c - a = 9.36 \qquad \log (c - a) = 0.9713$$
$$c + a = 81.36 \qquad \log (c + a) = 1.9104$$
$$\log (c^2 - a^2) = 2.8817$$

On the other hand

$$\log b^2 = 2 \log b = 2.8818$$

so that in applying the check we find a discrepancy of only one unit in the last place, and this amount of uncertainty is always present in logarithmic work, which is merely approx-

imative. Indeed in long pieces of computation or in cases where logarithms have been multiplied by numbers as great as 3 or 4 (so that errors are magnified) the last figure may be even more inaccurate than this.

Freedom from error in computations of all kinds depends much upon strict adherence to a uniform arrangement of the work. The student should at once adopt some definite plan and form the habit of using this plan for every triangle. Before beginning the use of the tables, he should have the whole work written out in skeleton form with blank spaces left where the values to be found in the tables can be entered.

An important part of any scheme which may be adopted is a reasonably accurate drawing of the triangle to be solved.

Example 2. Given $A = 30.57°$,
$c = 218.3$.
Find B, a, b.

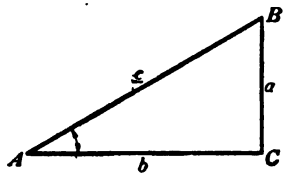


FIG. 11

$$\sin A = \frac{a}{c} \qquad \cos A = \frac{b}{c}$$

$$a = c \sin A \qquad b = c \cos A$$

$$= 218.3 \sin 30.57° \qquad = 218.3 \cos 30.57°$$

$$\log 218.3 = 2.3391 \qquad \log 218.3 = 2.3391$$

$$\log \sin 30.57° = \bar{1}.7064 \qquad \log \cos 30.57° = \bar{1}.9350$$

$$\log a = \overline{2.0455} \qquad \log b = \overline{2.2741}$$

$$a = \boxed{111.0} \qquad b = \boxed{188.0}$$

$$B = 90° - A = \boxed{59.43°}$$

CHAPTER III

FUNCTIONS OF ANGLES OF ANY MAGNITUDE AND OF THE SUM OF TWO ANGLES

16. Angles of Any Magnitude. An angle A may be measured by the amount by which a straight line through its vertex must be turned about this point in order to pass from the position AB of one of the sides of the angle to the position AB' of the other. The sides AB and AB' we speak of as the *initial* and *terminal* sides of the angle, respectively. The rotation takes place in the direction of the arrow in the figure. We might, however, equally well have taken AB' as the initial side, in which case the rotation would take place in the reverse direction. It is customary to select the direction of rotation indicated by the arrow in the figure as the positive direction of rotation. Rotations in the reverse direction will be spoken of as negative rotations. We are thus led to the conception of positive and negative angles.

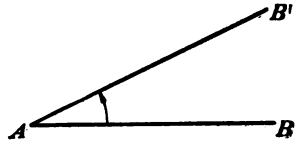


FIG. 12

We will agree that when we write the angle BAB' we mean that AB is to be the initial side. This is therefore the angle of the above figure. The negative angle would be indicated by $B'AB$. That is, *the side mentioned first in writing the angle shall always be the initial side.*

We may now consider angles of any magnitude, not only positive and negative acute angles (*i.e.* angles between 0° and + 90° or between 0° and − 90°) and positive and negative obtuse angles, but also angles, either positive or negative, containing more than 180° or even more than 360°. We have indicated in the figure a positive angle of this sort.

In the first positive rotation of 360°, the revolving line sweeps over the whole plane. The parts of the plane swept out between 0° and 90°, between 90° and 180°, between 180° and 270°, and between 270° and 360° are called the first, second, third, and fourth quadrants, respectively; and according as the terminal side lies in one or another of these quadrants, we say that we have an angle of the first, second, third, or fourth quadrant. Thus the angles 60°, 370°, − 290°, + 740° all lie in the first quadrant. The angles − 20°, + 300° lie in the fourth quadrant, etc.

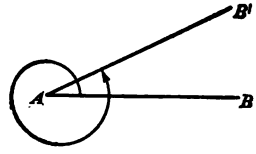


FIG. 13

17. Positive and Negative Segments. We are familiar from elementary algebra with the convention that segments measured along a straight line in one direction are regarded as positive, those measured in the opposite direction as negative.

In dealing with an angle BAB' we will agree that segments lying on the line AB (or AB produced) shall be regarded as positive if measured in the direction from A to B, as negative when measured in the opposite direction; and that segments on the line AB' (or AB' produced) shall be regarded as positive if measured in the direction from A to B', as negative when measured in the opposite direction. Moreover, when we speak of a segment MN, we shall always understand that it is measured from M to N; while if we wish to have it measured in the opposite direction, we write NM. In the figure, MN and LM are positive segments, NL is negative. The segment AB' is positive, $B'A$ is negative.

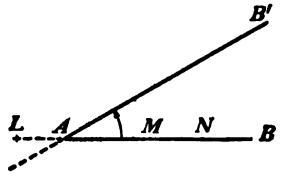


FIG. 14

We shall commonly place our angles as we have done in Figure 14, so that the initial line AB is horizontal and extends

to the right. We shall then make the further convention that all vertical segments shall be regarded as positive when measured upward, as negative when measured downward. We thus have, so far as horizontal and vertical directions go, exactly the convention with which we are familiar in the plotting of graphs in elementary algebra.

We must remember that on oblique lines no convention of sign is made except on the terminal side of the angle we are considering.

18. Functions of Angles of Any Magnitude. Let P be any point on the terminal side of the angle BAB'. From P drop a perpendicular on the initial side of the angle (or the initial side produced) meeting it in M. Then the primary trigonometric functions of the angle BAB' are defined as follows:

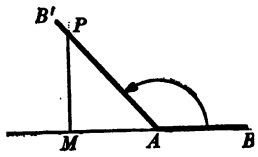


FIG. 15

$$\text{(1)} \qquad \begin{aligned} \sin BAB' &= \frac{MP}{AP}, \\ \cos BAB' &= \frac{AM}{AP}, \\ \tan BAB' &= \frac{MP}{AM}, \end{aligned}$$

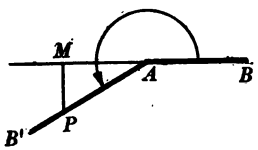


FIG. 16

where the segments AM, MP, AP are to be taken with the proper sign according to the conventions of § 17.

The secondary functions are, as in § 1, defined as the reciprocals of the primary functions:

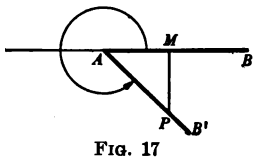


FIG. 17

$$\text{(2)} \qquad \begin{aligned} \operatorname{ctn} BAB' &= \frac{1}{\tan BAB'}, \\ \sec BAB' &= \frac{1}{\cos BAB'}, \\ \csc BAB' &= \frac{1}{\sin BAB'}. \end{aligned}$$

These definitions are perfectly general and apply to angles of any magnitude and in any quadrant.

Signs of the Functions. The trigonometric functions thus take on different signs according to the quadrant in which the angle lies. If the angle is in the first quadrant, all three segments AM, MP, AP are positive. Consequently all the functions are positive, and our present definitions reduce precisely to the definitions of Chapter I.

Any angle in the second quadrant (Fig. 18) has

a positive sine $\dfrac{MP}{AP} = \dfrac{+}{+}$,

a negative cosine $\dfrac{AM}{AP} = \dfrac{-}{+}$,

a negative tangent $\dfrac{MP}{AM} = \dfrac{+}{-}$.

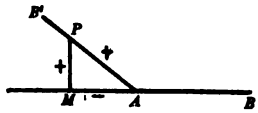


FIG. 18

For an angle in the third quadrant (Fig. 19) we have

a negative sine $\dfrac{-}{+}$,

a negative cosine $\dfrac{-}{+}$,

a positive tangent $\dfrac{-}{-}$,

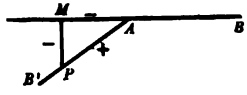


FIG. 19

while in the fourth quadrant the sine and tangent are negative, the cosine positive.

The signs of the secondary functions may be inferred at once from those of the primary functions by means of their definitions (2). Thus the secant is positive or negative according as the cosine is positive or negative; the sign of the cotangent is the same as the sign of the tangent, and the sign of the cosecant is the same as the sign of the sine.

The student will do well to make a little table indicating the signs of the six trigonometric functions for each of the four quadrants. Such a table should not, however, be committed to memory. The student should have the definitions

of the trigonometric functions and the conventions of sign so clearly in mind that he can see at a glance what sign a given function of a particular angle has. Only in this way can mistakes be avoided in the long run. Some students will find it convenient to use the line values explained in § 20 for remembering the signs of various functions.

Every angle greater than 360° and also every negative angle has its terminal and initial sides in exactly the same positions as some positive angle less than 360°. Hence the functions of an angle greater than 360° are equal to the same functions of the positive angle less than 360° whose sides coincide with those of the given angle; and a similar remark applies to the functions of negative angles.

Alternative Form of Definition. In conclusion we note that, if we prefer, we may always take P on the terminal side produced backwards. This has the effect of making AP negative instead of positive, but it also reverses the signs of both the other lines AM and MP, and thus leaves the various *ratios* of these lines unchanged. Thus we may replace Figure 15 by Figure 20 without changing any of the trigonometric functions. The angle BAB' is still an angle of the second quadrant.

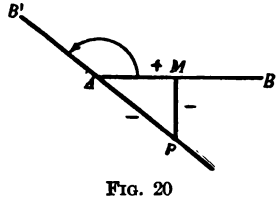


FIG. 20

19. Projections. Let us now suppose that we have two indefinite straight lines AB, CD, on each of which one direction is chosen as positive, as is indicated in the figure by the arrows. Let PQ be any segment on the

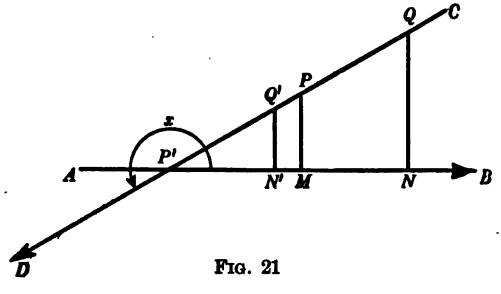


FIG. 21

line CD. (In the figure PQ is negative.) Its projection MN on AB may be found, as in § 7, by shifting PQ along CD until P falls at the intersection, P', of the two lines, and Q at Q'. If then we take for x the angle from the positive part of AB issuing from P' to the positive part of CD issuing from P' (in the figure x is an angle in the third quadrant) we have, as before, by § 18,

$$\cos x = \frac{P'N'}{P'Q'}$$

or

$$\overline{P'N'} = \overline{P'Q'} \cos x. \tag{1}$$

If the angle from the positive direction of a line AB to the positive direction of the line CD is x, then the projection of a segment, PQ, of CD (having regard to sign) on AB is equal to the product of PQ by the cosine of x.

On the other hand, $N'Q'$ is the projection of PQ on a line perpendicular to AB, whose positive direction we assume to make a *positive* right angle with the positive direction of AB. Hence, by precisely the same reasoning as that just used, we see that the projection of PQ on this perpendicular direction is

$$\overline{PQ} \sin x. \tag{2}$$

By the projection of a broken line whose ends are P and Q (thought of as starting at P and ending at Q) on the indefinite line AB is understood the segment MN of the line from the projection of P to the projection of Q.

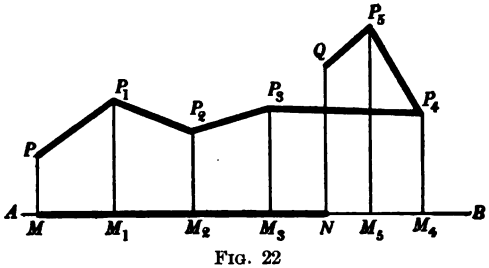


FIG. 22

The following fact is often useful: *The projection of a broken line is the sum of the projections of its parts.*

This is obviously the case if the broken line does not double back on itself; that is, if it constantly progresses to the right or to the left (the line AB being supposed horizontal). Even if it does double back on itself, as in the figure, a little reflection will show that it is still true when we take account of the algebraic signs. For instance, in the figure the projections of the first four parts are MM_1, M_1M_2, M_2M_3, M_3M_4, all positive, and these add up to MM_4. To this we must add the last two projections M_4M_5, M_5N, and, since these are negative, this amounts to a subtraction, and we have left precisely MN.

20. Line Values of Functions. About the vertex O of a given angle as center describe a circle of unit radius cutting the initial side in B and the terminal side in P_2 (Fig. 23). We suppose at first that the angle lies in the second quadrant. Draw M_2P_2 perpendicular to the diameter BB'. Then, by the definitions of § 18, taking account of the algebraic signs, we have

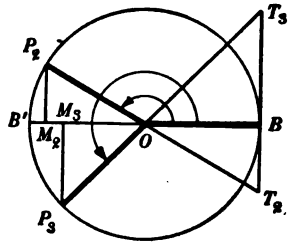


FIG. 23

$$\sin BOP_2 = \frac{M_2P_2}{OP_2} = \frac{M_2P_2}{1} = M_2P_2,$$

$$\cos BOP_2 = \frac{OM_2}{OP_2} = \frac{OM_2}{1} = OM_2.$$

Line values for the tangent and secant may be obtained by drawing at B a tangent to the circle and producing the terminal side, OP_2, backward till it meets this tangent in T_2.

We have then

$$\tan BOP_2 = \frac{BT_2}{OB} = \frac{BT_2}{1} = BT_2,$$

$$\sec BOP_2 = \frac{OT_2}{OB} = \frac{OT_2}{1} = OT_2.$$

To get line values for the cotangent and cosecant, we first draw a radius OC, so that $BOC = +90°$, and then draw the tangent at C. Let S_2 (Fig. 24) be the point where this tangent is met by the terminal side OP_2. Then

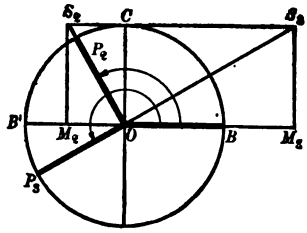


FIG. 24

$$\csc BOP_2 = \frac{OS_2}{M_2S_2} = \frac{OS_2}{OC} = \frac{OS_2}{1} = OS_2,$$

$$\operatorname{ctn} BOP_2 = \frac{OM_2}{M_2S_2} = \frac{CS_2}{OC} = \frac{CS_2}{1} = CS_2.$$

In precisely the same way, if we have an angle BOP_3 in the third quadrant, we see from Figure 23 that

$$\sin BOP_3 = M_3P_3, \qquad \tan BOP_3 = BT_3,$$
$$\cos BOP_3 = OM_3, \qquad \sec BOP_3 = OT_3.$$

and from Figure 24 that

$$\csc BOP_3 = OS_3, \qquad \operatorname{ctn} BOP_3 = CS_3.$$

The student should construct in the same way line values for the six trigonometric functions of an angle in the fourth quadrant.

For angles in the first quadrant, these line values are precisely those obtained in § 8.

It should be carefully noticed that, in constructing these line values, the tangents to the circle are always to be drawn at the points B and C, not at the left-hand end B' of the horizontal diameter, or at the lower end of the vertical diameter.

Variation of the Functions. In § 8 we saw how the functions change as the angle varies from 0° to 90°. In just the same way we can see, by using the line values, how the func-

tions change in any of the other quadrants. For instance, as the angle varies from 90° to 180°(Fig 23)

$$M_2P_2 = \sin A \text{ decreases from } +1 \text{ to } 0,$$
$$OM_2 = \cos A \text{ decreases from } 0 \text{ to } -1,$$
$$BT_2 = \tan A \text{ increases from } -\infty \text{ to } 0.$$

The student should consider in the same way the changes of the secondary functions in this quadrant, and also the changes of all the functions in the third and fourth quadrants.

21. Reduction to Positive Acute Angles. The trigonometric tables range only from 0° to 90°. Although they are not *directly* available for other angles, we shall now see how by a very simple method they can be made indirectly available. We illustrate this first in the case of a positive obtuse angle BAP. Since this angle is in the second quadrant, its sine is +, cosine −, tangent −, etc. It is, then, merely the numerical values of the various functions we still need to find. These are given by the ratios of the sides of the triangle APM, all these sides being now regarded as positive. That is, the trigonometric functions of the obtuse angle BAP are, except for sign, equal to the same functions of the positive acute angle PAB'. This angle we lay off in the position BAP', and we call it the acute angle *corresponding* to BAP. It is, in this case, merely the supplement: $BAP' = 180° - BAP$.

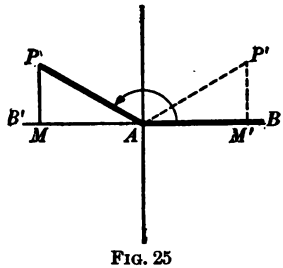


FIG. 25

For instance, suppose we want sin 160° and cos 160°. The angle *corresponding* to 160° is 20°. Consequently sin 160° $= \pm \sin 20°$, $\cos 160° = \pm \cos 20°$. But the sine of an angle in the second quandrant is +, its cosine −, while the sine and cosine of 20° are both +. Hence, finally,

$$\sin 160° = \sin 20° = + 0.3420,$$
$$\cos 160° = - \cos 20° = - 0.9397.$$

In general, if A has any magnitude, we construct a *corresponding* angle as the positive acute angle between the terminal side AP and the horizontal line BB'; and here too (see Figs. 26, 27) each trigonometric function of the original angle is equal to the same trigonometric function of the corresponding angle, except perhaps for sign. The signs are determined, as explained in § 18, by noticing in what quadrant the angle lies.

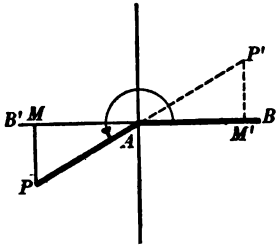


FIG. 26

Thus to find sin 300°. The terminal side of this angle makes an angle of 60° with the horizontal, hence sin 300° $= \pm$ sin 60°. Moreover 300° is in the fourth quadrant, hence (see § 18) its sine is negative; and

$$\sin 300° = - \sin 60° = - \tfrac{1}{2}\sqrt{3}.$$

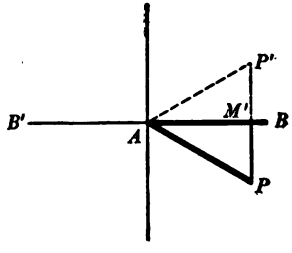


FIG. 27

22. The Functions of $A \pm 90°$, $A \pm 180°$, etc. The trigonometric functions of $A \pm 90°$, $A \pm 180°$, and of other similar expressions, can readily be expressed in terms of the functions of A. The formulas for doing this are far too numerous to be remembered, and we shall not do more than deduce a few as samples. The *method* we shall use should, however, be remembered, as it will enable the student at a moment's notice to deduce these or any similar formulas.

sin $(A + 180°)$. The terminal side of the angle $A + 180°$ is obviously the continuation of the terminal side of A. Consequently these two lines make the same angle with the horizontal, and their *corresponding* positive acute angles are

the same (see § 21). Hence their sines have the same numerical values

$$\sin(A + 180°) = \pm \sin A.$$

The negative sign must be used here because, if the terminal side of A lies below the horizontal line BB' (Fig. 26), the terminal side of $A + 180°$ lies above, and *vice versa*. Hence, finally

(1) $$\sin(A + 180°) = -\sin A.$$

$A + 90°$. A little less simple is the determination of $\sin(A + 90°)$ and $\cos(A + 90°)$.

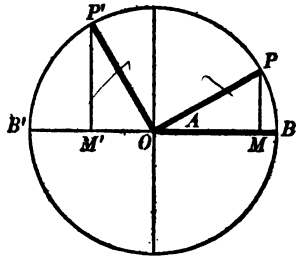


FIG. 28

Construct OP' so that $POP' = +90°$, P and P' both on the unit circle. Then $BOP' = A + 90°$. Now drop perpendiculars from P and P' on BB' meeting it in M and M'. The two triangles OMP and $P'M'O$ are equal; and since, except for sign, the vertical side of the second is $\sin(A + 90°)$, while the equal horizontal side of the first is $\cos A$, we have

$$\sin(A + 90°) = \pm \cos A.$$

By comparing the horizontal side of the second with the vertical side of the first, we find

$$\cos(A + 90°) = \pm \sin A.$$

Since, in the figure, $\cos A$ and $\sin(A + 90°)$ have the same sign, while $\sin A$ and $\cos(A + 90°)$ have opposite signs, we find at last

(2) $$\sin(A + 90°) = \cos A,$$

(3) $$\cos(A + 90°) = -\sin A.$$

We have proved these formulas only when A lies in the first quadrant. The student should draw the corresponding figures when A lies in the second, third, and fourth quad-

rants. He will then see that the reasoning we have used applies without change to these figures, so that formulas (2), (3) are true in all cases.

Functions of $-A$. Another similar question concerns the values of the trigonometric functions of the negative of an angle. Since two angles which are the negatives of one another evidently have the same *corresponding* positive acute angle (see § 21), it is clear that their trigonometric functions can differ at most in sign,

$$\sin(-A) = \pm \sin A, \qquad \cos(-A) = \pm \cos A, \text{ etc.}$$

The two angles A and $-A$ either lie one in the first and one in the fourth quadrants, or else one in the second and one in the third quadrants. In both cases the sine of one is positive, of the other negative, and the same is true of their tangents:

$$\sin(-A) = -\sin A, \tag{4}$$

$$\tan(-A) = -\tan A. \tag{5}$$

Similarly, we see that their cosines are either both positive or both negative:

$$\cos(-A) = \cos A. \tag{6}$$

23. The Fundamental Identities. The fundamental identities (2)–(5) of § 9, which we there established for positive acute angles, hold without change for angles of any magnitude. For, suppose A is any angle and A_1 the corresponding positive acute angle. Since $\sin A$ and $\cos A$ differ from $\sin A_1$, $\cos A_1$ at most in sign, their squares will not differ at all, and consequently, from the identity

$$\sin^2 A_1 + \cos^2 A_1 = 1,$$

which holds since A_1 is a positive acute angle, the identity

$$\sin^2 A + \cos^2 A = 1$$

follows at once. In precisely the same way the identities (3), (4) of § 9 follow for any angle.

To establish formula (5) of § 9 for any angle A, we refer to the definitions (1) of § 18. By dividing the first of these formulas by the second, we find

$$\frac{\sin A}{\cos A} = \frac{MP}{AP} \div \frac{AM}{AP} = \frac{MP}{AM},$$

and this, by the last of formulas (1), § 18, is $\tan A$.

Applications. One important question to which these formulas can be applied is: Given one trigonometric function of A, to determine the others. For instance, given $\sin A = -\frac{3}{4}$, find $\cos A$, $\tan A$. From (2), § 9, we have

$$\tfrac{9}{16} + \cos^2 A = 1,$$

or $$\cos^2 A = \tfrac{7}{16}. \quad \therefore \cos A = \pm \tfrac{1}{4}\sqrt{7}.$$

From (5), § 9, we have

$$\tan A = \frac{\sin A}{\cos A} = -\frac{3}{4} \div \left(\pm \frac{1}{4}\sqrt{7}\right) = \mp \frac{3}{\sqrt{7}} = \mp \frac{3}{7}\sqrt{7}.$$

It will be seen that the problem is not a completely determinate one, on account of the doubtful sign which is involved in the answers. Since $\sin A$ is given to us as a negative quantity, the angle A must be in either the third or fourth quadrant. The cosine is negative in the third quadrant, positive in the fourth. Hence we see that the upper sign corresponds to the case in which A is in the fourth quadrant, the lower sign to that in which it is in the third.

By the method here used, when any one of the functions is given, and also the quadrant in which the angle lies, we can always determine the other functions completely without the use of tables.

24. Circular Measure of Angles. In the higher parts of mathematics it is often convenient to take as a unit of angular measurement not the degree, but an angle called the *radian* and equal to a little more

than 57°. This angle may be defined accurately by laying off on any circle an arc equal in length to the radius of the circle. The angle subtended by this arc at the center of the circle is called a radian. Since this arc is contained $\pi = 3.142$ times in the semi-circle, we see that

$$\pi \text{ radians} = 180°,$$

so that, as was stated above, one radian is a little more than 57°. A table of trigonometric functions with the angles given in radians is given on the back cover of Huntington's Tables (unabridged).

25. Graphs of the Functions. The change in the values of the functions as the angle increases from 0° to 360° can be graphically represented by plotting the equations $y = \sin x$, $y = \cos x$, $y = \tan x$, etc., as curves.

These graphs may be constructed as follows: Set up two perpendicular axes such as are used in the plotting of algebraic equations. Let the horizontal one be the axis of x, and the vertical one the axis of y. In constructing these graphs, the angle x could be measured in degrees, but, for the purpose of showing the changes in the functions, a larger unit is better. The unit commonly adopted is the radian explained in § 24.

The equation $y = \sin x$ can now be plotted by giving various values to x (in radians) and computing the corresponding values of y from the table.

Another and more convenient method is to draw a circle with radius 1 to the left of the axis, with its center on the axis of x, and construct at the center of this circle angles at any convenient interval, for example, $\frac{\pi}{12}, \frac{2\pi}{12}, \ldots$. The vertical projections of the radii will be, as in § 20, the values of $\sin\frac{\pi}{12}, \sin\frac{2\pi}{12}$, etc.

If we now lay off on the axis of x the points $\frac{\pi}{12}, \frac{2\pi}{12}$, etc., remembering that the radius of the circle is our unit of length, and measure up or down from these points, as the case may be, distances equal to the corresponding vertical projections, we shall have points on the required graph. If we connect these points by a smooth curve, we shall have the Sine Curve of Figure 29.

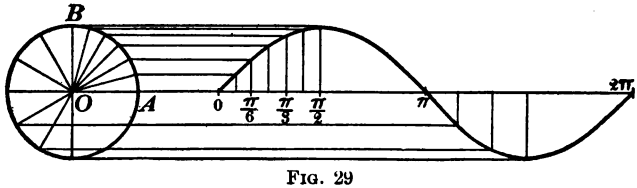


FIG. 29

The Cosine Curve is plotted in a similar manner, except that the angles should be measured from OB as the initial line instead of from OA. Thus the vertical projections on OB will now be the cosines, and we get the Cosine Curve of Figure 30.

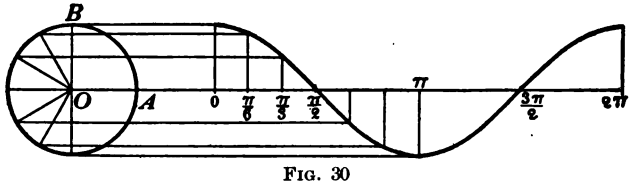


FIG. 30

The Tangent Curve also may be constructed by the method used for the sine curve, the line values of the tangent being constructed as in Figure 23 of § 20.

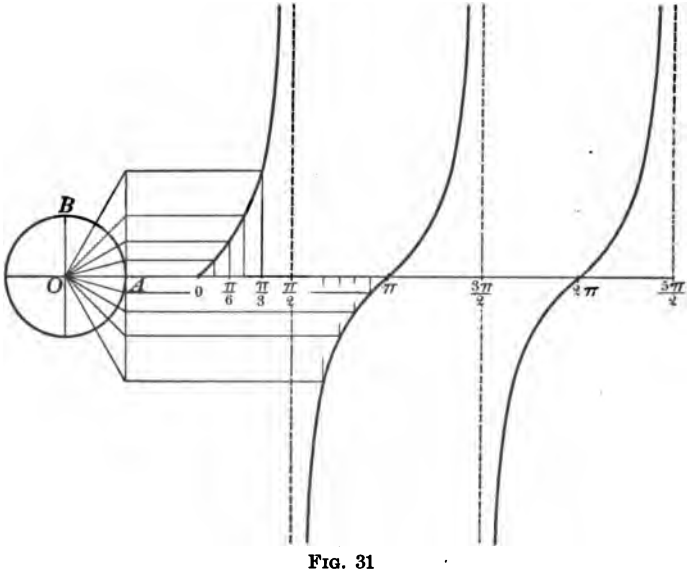


FIG. 31

26. Functions of the Sum and Difference of Two Angles. Let us try to express $\sin(x + y)$ and $\cos(x + y)$ in terms of $\sin x$, $\cos x$, $\sin y$, $\cos y$.

Starting with the horizontal radius OB, we lay off the angle $BOC = x$. With OC as initial side, we lay off

$COP = y$. Then $BOP = x + y$. We take, as usual, OB, OC, OP as the positive directions on these three lines. We consider also radii OB' and OC' making angles of $+90°$ with OB and OC, respectively, and we take, as usual, OB' and OC' as the positive directions on these lines. From P we drop a perpendicular PN on OC. We shall find that the formulas we want come immediately by taking the horizontal and vertical projections of the broken line ONP.

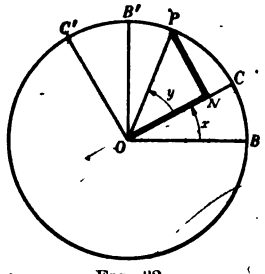


FIG. 32

Since $OP = 1$, we have, by (1) and (2), § 19,

$$ON = \cos y, \qquad NP = \sin y.$$

By the same formulas and formulas (2), (3), § 22,

(1)
$$\begin{cases} \text{Horizontal projection of } ON = ON \cos x = \cos y \cos x, \\ \text{Vertical projection of } ON = ON \sin x = \cos y \sin x, \\ \text{Horizontal projection of } NP = NP \cos(x + 90°) \\ \qquad\qquad = -\sin y \sin x, \\ \text{Vertical projection of } NP = NP \sin(x + 90°) \\ \qquad\qquad = \sin y \cos x. \end{cases}$$

Now by the theorem at the end of § 19

(2)
$$\begin{cases} \text{Vert. proj. } OP = \text{vert. proj. } ON + \text{vert. proj. } NP, \\ \text{Horiz. proj. } OP = \text{horiz. proj. } ON + \text{horiz. proj. } NP. \end{cases}$$

Replacing the first members of (2) by their values $\sin(x + y)$ and $\cos(x + y)$, respectively, and replacing their second members by their values from (1), we find

(3) $$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

(4) $$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

The proof here given of these very important formulas is perfectly general, applying no matter what the magnitudes

or signs of the angles x and y may be. The student should convince himself of this fact by drawing a variety of figures and seeing that the proof applies without any change whatever to them all. One such figure is given here (Fig. 33).

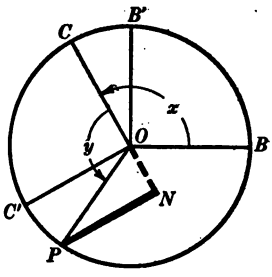


FIG. 33

The formula for $\tan(x+y)$ is derived by applying the above values in the equation (5) of § 9. This gives us

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

In order to change the functions on the right to $\tan x$ and $\tan y$, we divide numerator and denominator by $\cos x \cos y$. Then, by (5), § 9, the formula reduces to

$$(5) \qquad \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Since these formulas are true for any angles x and y, we may replace y by $(-y)$ and obtain formulas for the functions of $(x-y)$. Thus we have

$$\sin(x-y) = \sin x \cos(-y) + \cos x \sin(-y).$$

This may be reduced to a simpler form by (4), (6), § 22, giving

$$(6) \qquad \sin(x-y) = \sin x \cos y - \cos x \sin y.$$

In the same way we find

$$(7) \qquad \cos(x-y) = \cos x \cos y + \sin x \sin y,$$

$$(8) \qquad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

27. Functions of $2x$ and $\frac{1}{2}x$. If we let $y = x$, formulas (3), (4) of § 26 become

$$\sin(x+x) = \sin x \cos x + \cos x \sin x,$$
$$\cos(x+x) = \cos x \cos x - \sin x \sin x,$$

or

$$\sin 2x = 2 \sin x \cos x, \tag{1}$$

$$\cos 2x = \cos^2 x - \sin^2 x. \tag{2}$$

If, in this last formula, we replace $\sin^2 x$ by its value $1 - \cos^2 x$, or $\cos^2 x$ by $1 - \sin^2 x$, we find two other forms for (2)

$$\cos 2x = 2\cos^2 x - 1, \tag{3}$$

$$\cos 2x = 1 - 2\sin^2 x. \tag{4}$$

From (5), § 26, we find, by letting $y = x$,

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \tag{5}$$

Since (4) is true for every angle x, we may write $x = \frac{1}{2}y$. Formula (4) then becomes, after an obvious slight reduction

$$\sin \frac{y}{2} = \pm\sqrt{\frac{1 - \cos y}{2}}. \tag{6}$$

Similarly from (3) by letting $x = \frac{1}{2}y$, we find

$$\cos \frac{y}{2} = \pm\sqrt{\frac{1 + \cos y}{2}}. \tag{7}$$

The proper choice of sign in (6) and (7) can be made only when we know in which quadrant the angle $\frac{1}{2}y$ lies.

By dividing (6) by (7) we obtain at once a formula for $\tan(\frac{1}{2}y)$ which also contains a doubtful sign. A better formula may, however, be obtained by the following device:

$$\tan \frac{y}{2} = \frac{\sin \frac{y}{2}}{\cos \frac{y}{2}} = \frac{2 \sin^2 \frac{y}{2}}{2 \sin \frac{y}{2} \cos \frac{y}{2}}.$$

This numerator is seen, by (6), to be equal to $1 - \cos y$, while, if in (1) we let $x = \frac{1}{2} y$, we see that the denominator is equal to $\sin y$. Hence

$$\tan \frac{y}{2} = \frac{1 - \cos y}{\sin y}. \tag{8}$$

By a precisely similar method we find as an alternative form

$$\tan \frac{y}{2} = \frac{\sin y}{1 + \cos y}. \tag{9}$$

28. Sums and Differences of Sines and Cosines. By adding and subtracting formulas (3) and (6), and also formulas (4) and (7), of § 26 we find

$$\begin{cases} \sin (x+y) + \sin (x-y) = 2 \sin x \cos y, \\ \sin (x+y) - \sin (x-y) = 2 \cos x \sin y, \\ \cos (x+y) + \cos (x-y) = 2 \cos x \cos y, \\ \cos (x+y) - \cos (x-y) = -2 \sin x \sin y. \end{cases} \tag{1}$$

These formulas enable us to express the product of the sine or cosine of one angle by the sine or cosine of another as the sum of sines or cosines of two other angles. This is sometimes useful. At other times, however, for instance when we wish to perform a logarithmic computation, we wish to do just the reverse and to replace the sum or difference of two sines or of two cosines by a product. Formulas for this purpose are obtained from (1) by letting $x = \frac{1}{2}(A + B)$, $y = \frac{1}{2}(A - B)$. Formulas (1) then become

$$\begin{cases} \sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B), \\ \sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B), \\ \cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B), \\ \cos A - \cos B = -2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B). \end{cases} \tag{2}$$

29. Trigonometric Identities. We come back once more to the question, treated at the close of § 9, of establishing new identities. Everything there said holds in the more general case we are now considering in which the angles used may have any magnitudes. In establishing new identities, however, we must be prepared to use not merely the fundamental identities of § 9 and the definitions of the secondary trigono-

metric functions of § 1, but also the more recently derived identities of §§ 26, 27 (and also, if we like, of § 28, though these latter, being merely consequences of the formulas of § 26, are never indispensable).

Example 1. $\dfrac{1+\sin 2x}{1-\sin 2x} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2$.

$$\frac{1+2\sin x\cos x}{1-2\sin x\cos x} \;\Big|\; \frac{1+2\tan x+\tan^2 x}{1-2\tan x+\tan^2 x}$$

$$\Big|\; \frac{1+\dfrac{2\sin x}{\cos x}+\dfrac{\sin^2 x}{\cos^2 x}}{1-\dfrac{2\sin x}{\cos x}+\dfrac{\sin^2 x}{\cos^2 x}}$$

$$\Big|\; \frac{\cos^2 x+2\sin x\cos x+\sin^2 x}{\cos^2 x-2\sin x\cos x+\sin^2 x}.$$

The arrangement here adopted is often a convenient one. On the right of the vertical bar we have successive reductions of the right-hand member of the identity to be established; the first reduction being performed here by simply performing the operation of squaring; the second, by replacing $\tan x$ by $\dfrac{\sin x}{\cos x}$; the third, by multiplying numerator and denominator by $\cos^2 x$. The left-hand member we reduce in a similar way on the left of the vertical bar. Here only one such reduction is made, namely, by replacing $\sin 2x$ by its value $2\sin x\cos x$. In this process we must constantly compare the results we get on the two sides of the vertical bar, remembering that our identity will be proved as soon as we can identify the expressions on the two sides. We must then be constantly trying to make them more alike. This was the reason, in this case, for replacing $\tan x$ on the right by its value in terms of $\sin x$ and $\cos x$, since it is these functions and not the tangent which occur on the left. The final expression on the right is obviously equal to that on the left on account of the relation $\sin^2 x+\cos^2 x=1$.

Example 2. $\dfrac{1-\cos x}{\sin x} = \dfrac{\sin x}{1+\cos x}$.

In this identity there is no very obvious method of simplifying either side. In such cases it is sometimes better, instead of trying directly to prove the two members equal, to take their difference and prove that this is identically zero. In this case the steps would be

$$\frac{1-\cos x}{\sin x} - \frac{\sin x}{1+\cos x} = \frac{1-\cos^2 x - \sin^2 x}{\sin x\,(1+\cos x)}.$$

Since the numerator of this fraction is identically zero (formula (2), § 9), the fraction is zero, and our identity is established.

30. Trigonometric Equations. Referring back to § 10, we see that what we were there doing may now be described as finding the positive acute-angle solutions of the equations in question. The equations have many other solutions besides these, and when we are asked to solve a trigonometric equation, it will, from now on, be understood that we are to find *all* of its solutions.

Example 1. $\sin x = \frac{1}{2}$.

Since $\sin x$ is to be positive, x must lie in the first or second quadrant. The only possible acute angle is 30°. The only possible angle between 90° and 180° is the one which has 30° as its *corresponding* angle, that is, 150°. We may, however, increase or diminish these angles by 360° without affecting their sine; or, more generally, we may increase or diminish them by any multiple of 360°. Thus we see that our equation has an infinite number of solutions, all these solutions being given by the formulas

$$x = 30° + n\,360°,$$
$$x = 150° + n\,360°,$$

where n may take on all integral values, positive, negative, or zero.

Example 2. $2\sin^2 x - 3\sin x + 1 = 0.$

As in § 10, we see that this equation is satisfied when and only when one of the two equations

$$\sin x = 1, \qquad \sin x = \tfrac{1}{2}$$

is satisfied. The first of these has as its solution

$$x = 90° + n\,360°;$$

the second

$$x = 30° + n\,360°,$$
$$x = 150° + n\,360°.$$

Consequently these three formulas together give the complete solution of our equation.

Example 3. $20\cos y - 15\sin y = 12.$

The most obvious method would be to replace $\cos y$ by its value

$$\pm\sqrt{1 - \sin^2 y}.$$

We will explain instead a shorter method of solving the equation which depends on a device which will frequently be found useful. The first member of the equation is something like the second member of the formula (6), § 26 for $\sin(x - y)$: if we could only determine an angle x such that $\sin x = 20$, $\cos x = 15$, it would be precisely equal to it. This is impossible, since $\sin x$ and $\cos x$ can never be greater than 1. However, *this* difficulty may be obviated by dividing the equation through, before we begin, by a sufficiently large constant, for instance, 100. Our question then is whether we can find an angle x such that $\sin x = .20$, $\cos x = .15$. These quantities, being less than 1, may perfectly well be the values of a sine and of a cosine, but they are not the sine and cosine of *the same* angle, since the sum of their squares is not 1. Instead of dividing by 100 let us then divide by p, and try to choose this quantity so that

$$\left(\frac{20}{p}\right)^2 + \left(\frac{15}{p}\right)^2 = 1.$$

This gives $p^2 = 625$.

Let us then divide our equation by 25, thus writing it

$$\frac{4}{5}\cos y - \frac{3}{5}\sin y = \frac{12}{25}.$$

It is now easy to choose x so that

$$\sin x = \frac{4}{5}, \quad \cos x = \frac{3}{5};$$

all we have to do is to let $x = 53.13°$, as we see from the tables. Hence our equation may be written:

$$\sin (53.13° - y) = \frac{12}{25} = .4800.$$

Hence $\quad 53.13° - y = 28.69° + n\,360°,$

or $\quad 53.13° - y = 151.31° + n\,360°.$

Consequently, the complete solution of our equation is

$$y = 24.44° + m\,360°,$$
$$y = -\,98.18° + m\,360°.$$

One sometimes needs not all the solutions of an equation, but merely those of a certain range of magnitude; for example, if the angle which we are seeking is the angle of a triangle, we need not consider any angles except those between 0° and 180°. As an illustration we consider

Example 4. Find all positive angles less than 180° which satisfy the equation

$$\cos 3x = \tfrac{1}{2}.$$

Here either $\quad 3x = 60° + n\,360°,$

or $\quad 3x = -\,60° + n\,360°.$

Consequently the *complete* solution of our equation is

$$x = \pm\, 20° + n\,120°.$$

Using the upper sign, we get angles of the sort we want by letting $n = 0, 1$; using the lower sign, only by letting $n = 1$. Consequently there are three, and only three, answers

$$x = 20°,\ 100°,\ 140°.$$

31. Anti-trigonometric Functions. We have used the notation $\sin^2 x$, $\sin^3 x$, etc., to mean the square of $\sin x$, the cube of $\sin x$, etc. There is, however, one important ex-

ception to this notation: $\sin^{-1} x$ is not understood to mean the minus first power (or reciprocal) of $\sin x$, but is read *the anti-sine of* x, and means the angle whose sine is x. For instance, 30° would be the anti-sine of $\frac{1}{2}$, since $\sin 30° = \frac{1}{2}$. Similarly $\cos^{-1} x$ is read the anti-cosine of x and means the angle whose cosine is x; $\tan^{-1} x$, or the anti-tangent of x, means the angle whose tangent is x, etc. Thus

$$\sin^{-1} \tfrac{1}{2} = 30°, \quad \cos^{-1} \tfrac{1}{2} = 60°, \quad \tan^{-1} \tfrac{1}{2} = 26.56°.$$

It should be remembered, however, that there are an infinite number of angles with a given sine, and consequently, when x is given, there are an infinite number of values which we may take for $\sin^{-1} x$. For instance, instead of writing $\sin^{-1} \frac{1}{2} = 30°$ we may equally well write $\sin^{-1} \frac{1}{2} = 150°$, or 390°, or 510°, etc. A similar remark applies to all the other anti-trigonometric functions, or inverse trigonometric functions as they are sometimes called. This indeterminateness is a real source of difficulty in the use of the anti-trigonometric functions, but in cases in which there is no doubt as to the quadrant in which the angle lies, the notation is often a very convenient one.

CHAPTER IV

SOLUTION OF OBLIQUE TRIANGLES

In this chapter the angles used, being angles of a plane triangle, are less than 180°. We shall always consider them to be positive. Moreover all the lengths with which we deal shall be regarded as positive, and no distinction shall be made here between the notations AB and BA for a segment.

32. The Law of Sines. Let A and B be any two angles of a plane triangle. They may be both acute (Fig. 34), or one may be acute and the other obtuse (Figs. 35, 36). It is of no consequence in our reasoning whether the remaining angle, C, is acute or obtuse.

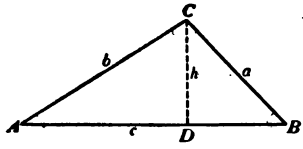


FIG. 34

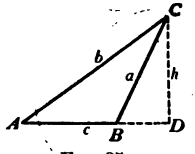


FIG. 35

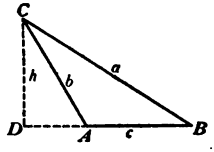


FIG. 36

In Figure 34 the triangles ADC and BDC may serve as triangles of reference for the angles A and B, and we have

$$h = b \sin A, \quad h = a \sin B.$$

The same is seen to be true in Figures 35 and 36, provided we recall that the sine of an obtuse angle is equal to the sine of its supplementary acute angle. Consequently in all cases

$$b \sin A = a \sin B,$$

or

$$\frac{\sin A}{\sin B} = \frac{a}{b}. \tag{1}$$

That is

THE LAW OF SINES. *Any two sides of a plane triangle are to each other as the sines of the opposite angles.*

We might, of course, write down a proportion similar to (1) for each pair of sides. More symmetrically, we may write the continued proportion

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{2}$$

33. The Law of Cosines. The Law of Sines has just been obtained by equating two different expressions for the altitude h. By doing this same thing in a different way, we shall now obtain a second important law. We have in Figure 34

$$\begin{cases} h^2 = b^2 - \overline{AD}^2, \\ h^2 = a^2 - \overline{BD}^2. \end{cases} \tag{1}$$

Equating these expressions, and replacing AD by its value $c - BD$, gives

$$b^2 - (c - BD)^2 = a^2 - \overline{BD}^2.$$

Hence

$$b^2 = a^2 + c^2 - 2\,c\overline{BD}. \tag{2}$$

But $$BD = a \cos B.$$

Consequently (2) may be written

$$b^2 = a^2 + c^2 - 2\,ac \cos B. \tag{3}$$

This same reasoning may be applied to Figure 36 with only the slight change that now AD must be replaced by $BD - c$. Since the square of this quantity is precisely the same as the square of $c - BD$, this change makes no difference in the final result. In Figure 35 the changes are slightly greater. We have equations (1), as before ; but now

$$AD = c + BD,$$

and substituting this value leads us not to (2) but to

$$b^2 = a^2 + c^2 + 2\,c\overline{BD}. \tag{4}$$

There is, however, also a second change; for since B is obtuse, $\cos B$ is negative, and $BD = -a \cos B$. The substitution of this in (4) leads us again to (3). Thus we see that (3) is true in all cases.

THE LAW OF COSINES. *The square of any side of a plane triangle is equal to the sum of the squares of the other two sides minus twice their product times the cosine of the included angle.*

This may be regarded as a generalization of the Pythagorean Theorem to which it reduces when the included angle is a right angle.

These two laws are among the most important of trigonometry.

34. The Law of Tangents. A third law of a somewhat similar nature, but far less important, is the Law of Tangents. It may be deduced as follows:

By first adding and then subtracting 1 from both sides of the equation (1), § 32, we get the two equations

$$\frac{a \pm b}{b} = \frac{\sin A \pm \sin B}{\sin B},$$

and by dividing one of these equations by the other, we find

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

By § 28, the right-hand side equals

$$\frac{2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)}{2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)} = \operatorname{ctn} \tfrac{1}{2}(A + B) \tan \tfrac{1}{2}(A - B).$$

Remembering that the cotangent is, by definition, the reciprocal of the tangent, we thus find

$$\frac{a - b}{a + b} = \frac{\tan \tfrac{1}{2}(A - B)}{\tan \tfrac{1}{2}(A + B)}, \tag{1}$$

a law whose content could easily be enunciated in words and is called *The Law of Tangents.*

35. Graphical Solution of Triangles. A triangle has six parts, the three sides and the three angles. The relation be-

tween the three angles is, however, so simple (namely, that their sum is 180°) that we may practically regard the triangle as having only five parts, its three sides and any two of its angles. The problem of solving a triangle is this: *any three* of these five parts being given, to determine the other two (more accurately the other three). It is convenient to distinguish the following four cases, which evidently exhaust all possibilities:

Case I. Given two angles and a side.

Case II. Given two sides and the angle opposite one of them.

Case III. Given two sides and the included angle.

Case IV. Given three sides.

Cases I, III, IV. All of these cases can easily be treated graphically by the methods of elementary plane geometry. For instance, in Case III we construct the given angle, and along its sides lay off from its vertex lengths equal to the two given sides. By connecting the points thus reached, we have clearly formed the only possible triangle with the three given parts; and by measuring the other parts, we get a graphical solution of the problem. Since this construction is always possible, we see that in this case, no matter what data are given us (we assume that all angles given are less than 180°), the problem has one and only one solution.

Similarly we can construct the triangle in Case IV from the three given sides, provided that one of the sides is given as less than the sum and greater than the difference of the other two; while if these conditions are not fulfilled, the problem has no solution.

In Case I it is necessary that the sum of the two given angles be less than 180°. If this condition is fulfilled, we find the third angle from the fact that the sum of the three equals 180°, and we then merely have to construct a triangle given two angles and the included side.

In Cases I, III, IV, then, we see that, apart from certain cases which are obviously impossible, the problem always has just one solution, and this solution may be readily found graphically.

The Ambiguous Case II. Suppose that the given parts are the angle A, the opposite side a, and another side, b. Suppose first that A is obtuse (Fig. 37): It is clear that unless a is greater than b, no triangle is possible. Suppose $a > b$. In this case there is one and only one triangle, which may be constucted as follows: Construct the given angle A, and on one side of this angle lay off AC equal to the given side b. With C as center and the given side a as radius, draw an arc of a circle cutting the other side of the angle A in B. ABC is the desired triangle.

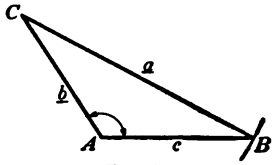


FIG. 37

Suppose, on the other hand, that the given angle A is acute (Fig. 38). Then the construction just described may lead us to two triangles, AB_1C and AB_2C. However, if the given side a is shorter than the perpendicular DC, the circle described about C as center will not meet the line AD, and the problem has no solution; while if it is just as long as CD, the problem has just one solution. On the other hand, a may be so long as to bring B_2 to the left of A, in which case the triangle AB_2C no longer has as its angle the given acute angle A, but its supplement, and AB_1C is the only solution of the problem. This case, $a > b$ (and obviously also the case $a = b$), therefore always has just one solution.

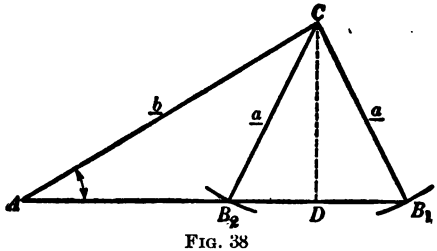


FIG. 38

We see, then, that in Case II we may have no solution, one solution, or two solutions, according to circumstances.

It follows from this discussion that in all four cases, I, II, III, IV, the drawing of a very rough sketch will enable us *without the memorizing of any rules* to see just how many solutions (0 or 1) our problem has except in a special case of Case II (the *Doubtful Case*), viz. the given angle acute and the opposite side less than the adjacent side. Here it may require a more accurate construction, or some other accurate method, to decide whether we have

$$b > a > DC, \text{ two solutions,}$$
$$a = DC, \text{ one solution,}$$
$$a < DC, \text{ no solution.}$$

The graphical method of solving triangles described in this section (or rather recalled to the reader, since it was already known to him from plane geometry) may be made with a little care, and with ordinary drawing instruments, to yield fairly accurate results. For greater accuracy the method of trigonometry described in the next section should be used. We shall find that the Law of Sines enables us to treat satisfactorily Cases I and II, while the Law of Cosines is all we need for Cases III and IV. In all cases, before beginning the trigonometric work, a careful sketch should be made, both to make sure whether the problem is a possible one, and to get a check against gross blunders in the numerical work.

36. Case I. Two Angles and One Side.

Given: $A = 52.16°$
$C = 80.52°$
$c = 14.26$

$B = 180° - (A + C) = 180° - 132.68°$
$= \boxed{47.32°}$

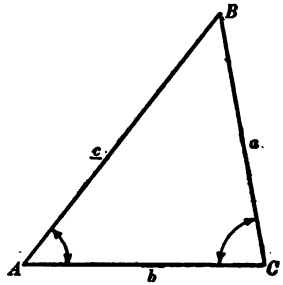


FIG. 39

By the Law of Sines

$$\frac{a}{c} = \frac{\sin A}{\sin C} \qquad \frac{b}{c} = \frac{\sin B}{\sin C}$$

$$a = \frac{c \sin A}{\sin C} \qquad b = \frac{c \sin B}{\sin C}$$

$\log c = 1.1541$	$\log c = 1.1541$
$\log \sin A = \bar{1}.8975$	$\log \sin B = \bar{1}.8663$
$\operatorname{colog} \sin C = \underline{0.0060}$	$\operatorname{colog} \sin C = \underline{0.0060}$
$\log a = 1.0576$	$\log b = 1.0264$
$a = \boxed{11.42}$	$b = \boxed{10.63}$

Check by the Law of Sines * in the form $a \sin B = b \sin A$.

No rule need be remembered here. One has merely to apply the Law of Sines in such a way that only one unknown part occurs each time. In the check we apply the Law of Sines (or the Law of Tangents) in such a way as to involve the two computed sides.

37. Case II. Two Sides and the Angle Opposite One of Them.

Given: $A = 121.35°$
$a = 145.0$
$b = 112.0$

Drawing a figure here (Fig. 40), we see that there is just one solution.

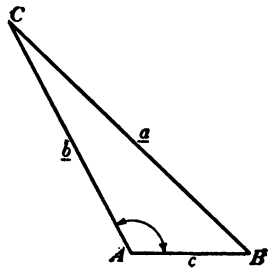


FIG. 40

* A more complete check is given by that form of the Law of Tangents which involves a and b. In this particular example the check by the Law of Tangents is not accurate since a and b are so nearly equal that $a - b$ is known to only two significant figures.

By the Law of Sines

$$\sin B = \frac{b \sin A}{a}$$

$$\log b = 2.0492$$
$$\log \sin A = \bar{1}.9315$$
$$\operatorname{colog} a = \bar{3}.8386$$
$$\log \sin B = \bar{1}.8193$$
$$B = \boxed{41.28^\circ}$$

So far as the analytic work goes, we should have the choice between two values for B, 41.28° and 138.72°, since these two angles have the same sine. From our figure (or from the fact that A is obtuse) we see that B must be acute, and the obtuse value for B must be discarded. Then

$$C = 180^\circ - (A + B) = 180^\circ - 162.63^\circ = \boxed{17.37^\circ}$$

The rest of the solution is performed by the Law of Sines,

$$c = \frac{b \sin C}{\sin B},$$

precisely as in Case I. The final solution is

$$B = 41.28^\circ, \qquad C = 17.37^\circ, \qquad c = 50.69.$$

The Doubtful Case; General Discussion. We will next give an example in which A is acute and $a < b$, a case in which, as we saw in § 35, we may have 0, 1, or 2 solutions. The length of the perpendicular DC (Fig. 38) is $b \sin A$, and accordingly we have by § 35

(1) no solution when $a < b \sin A$,
(2) one solution when $a = b \sin A$,
(3) two solutions when $a > b \sin A$,

and it would be a simple matter, when the numerical data are given, to find out which of these cases we have. It is, however, not necessary to spend time in doing this; for the

formula we shall use in solving the triangle is the Law of Sines

$$\sin B = \frac{b \sin A}{a}.$$

Consequently case (1), (2), or (3) will occur according as the fraction on the right has a value > 1, $= 1$, < 1. The first of these cases is obviously an impossible one, since it requires us to find an angle B whose sine is greater than 1. The second gives us only the one solution, $B = 90°$; while the third gives a value for $\sin B$ less than 1, and hence two values, one acute and one obtuse, for B, and both of these, as we see from (3), are actual solutions. Hence, instead of remembering the rules contained in the statements (1), (2), (3), we simply go through the numerical work of solution, and the fact that there is no solution, or only one solution, will show itself there. This, however, is true only in the doubtful case, *i.e.* the case in which a rough sketch leaves us in doubt as to how many solutions there are. In other cases, for instance in the numerical problem solved at the beginning of this section, the analytic work gives us, at this stage, no clue as to whether there are one or two solutions.

The Doubtful Case; Numerical Example.

Given: $A = 27.24°$
$a = 309.0$
$b = 360.0$

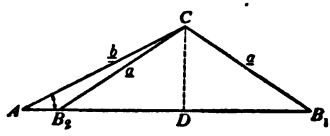


FIG. 41

$$\sin B = \frac{b \sin A}{a}$$

$\log b = 2.5563$
$\log \sin A = \bar{1}.6606$
$\operatorname{colog} a = \bar{3}.5100$
$\log \sin B = \bar{1}.7269$
$B_1 = \boxed{32.22°}$
$B_2 = 180° - 32.22° = \boxed{147.78°}$

$$C_1 = 180° - (A + B_1) = 180° - 59.46° = \boxed{120.54°}$$

$$C_2 = 180° - (A + B_2) = 180° - 175.02° = \boxed{4.98°}$$

$$c_1 = \frac{a \sin C_1}{\sin A} \qquad c_2 = \frac{a \sin C_2}{\sin A}$$

$$\begin{aligned} \log a &= 2.4900 \\ \log \sin C_1 &= \bar{1}.9351 \\ \operatorname{colog} \sin A &= 0.3394 \\ \log c_1 &= 2.7645 \\ c_1 &= \boxed{581.4} \end{aligned} \qquad \begin{aligned} \log a &= 2.4900 \\ \log \sin C_2 &= \bar{2}.9386 \\ \operatorname{colog} \sin A &= 0.3394 \\ \log c_2 &= 1.7680 \\ c_2 &= \boxed{58.61} \end{aligned}$$

In solving numerical problems under Case II, all we need remember is that the Law of Sines is to be used in such a way as to involve only one unknown part each time, and only those solutions are to be discarded which the sketch shows to be inapplicable. Of course the sketch should be drawn first, since it may show that the problem has no solution, in which case no numerical work will be necessary.

38. Case III. Two Sides and the Included Angle.

Given: $a = 17.00$
$b = 12.00$
$C = 21.33°$

$$c = \sqrt{a^2 + b^2 - 2ab \cos C}$$
$$= \sqrt{289.0 + 144.0 - 380.1}$$
$$= \sqrt{52.9}$$
$$c = \boxed{7.27}$$

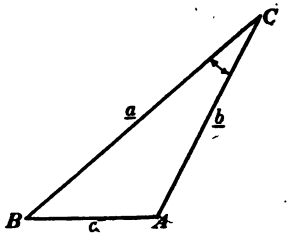


FIG. 42.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{144.0 + 52.9 - 289.0}{174.5}$$
$$= \frac{-92.1}{174.5} = -0.528$$
$$A = 180° - 58.1° = \boxed{121.9°}$$
$$B = 180° - (A + C) = 180° - 143.2° = \boxed{36.8°}$$

Check by the Law of Sines in the form $c \sin A = a \sin C$.

The angle A might have been obtained by use of the Law of Sines in the form just written. Using logarithms, we find in this way

$$\log \sin A = \bar{1}.9297,$$

the last figure here, however, being valueless since c is known to only three significant figures. We thus get two values for A, 58.3° and 121.7°. Of these, we must choose the second, since $a^2 > (b^2 + c^2)$ and consequently A is obtuse.

The method just explained has the disadvantage that the work is not well adapted to logarithmic computation. Nevertheless, if the given numbers, a and b, are such that their squares can be easily found, the method is a good one. It is particularly good if only the side c is desired, not the angles A and B.

The Right Triangle Method. This objection may, to a considerable extent, be obviated by the following method, which consists in reducing the solution of the proposed oblique triangle to the successive solutions of two right triangles. These right triangles are formed by dropping a perpendicular from the vertex of one of the unknown angles on the opposite side. This perpendicular forms two right triangles of which the given triangle is either the sum or the difference. One of these triangles (ADC in the numerical example which follows) can be immediately solved, and then the solution of the other is easy.

Given: $b = 55.12$
$c = 39.90$
$A = 94.39°$

Drop a perpendicular, CD, from C to AB produced. (If A were acute, D would lie on AB itself, and we

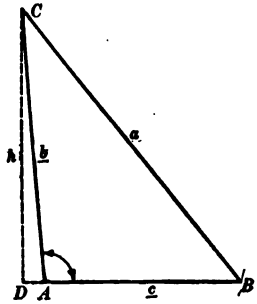


FIG. 43

should have $c - DA = DB$, otherwise all the following work would be unchanged.)

$$DA = b \cos DAC, \quad CD = b \sin DAC, \tan B = \frac{CD}{DB}$$

$$\begin{aligned} \log b &= 1.7414 \\ \log \cos DAC &= \bar{2}.8839 \\ \log DA &= \overline{0.6253} \\ DA &= 4.220 \\ DA + c = DB &= 44.12 \end{aligned} \qquad \begin{aligned} \log b &= 1.7414 \\ \log \sin DAC &= \bar{1}.9987 \\ \log CD &= \overline{1.7401} \\ \log DB &= 1.6446 \\ \log \tan B &= \overline{0.0955} \\ B &= \boxed{51.25^\circ} \end{aligned}$$

$$C = 180^\circ - (A + B) = 180^\circ - 145.64^\circ = \boxed{34.36^\circ}$$

$$\frac{CD}{a} = \sin B$$

$$a = \frac{CD}{\sin B}$$

$$\begin{aligned} \log CD &= 1.7401 \\ \log \sin B &= \bar{1}.8920 \\ \log a &= 1.8481 \\ a &= \boxed{70.48} \end{aligned}$$

Check by the Law of Sines in the form $a \sin C = c \sin A$.

Solution by the Law of Tangents. A neater solution, also well adapted for logarithmic computation, but which involves no less numerical work than the solution just given, is by means of the Law of Tangents.

$$\frac{b - c}{b + c} = \frac{\tan \frac{1}{2}(B - C)}{\tan \frac{1}{2}(B + C)}.$$

We explain this method by means of the same numerical example as was used to illustrate the last method. We first compute $b - c$, $b + c$, $\frac{1}{2}(B + C)$:

$b - c = 15.22$; $b + c = 95.02$; $\frac{1}{2}(B + C) = \frac{1}{2}(180° - A) = 42.80°$.

$$\tan \tfrac{1}{2}(B - C) = \frac{(b - c)\tan \tfrac{1}{2}(B + C)}{b + c} \qquad a = \frac{c \sin A}{\sin C}$$

$\log (b - c) = 1.1824$	$\log c = 1.6010$
$\log \tan \frac{1}{2}(B + C) = \bar{1}.9666$	$\log \sin A = \bar{1}.9987$
$\operatorname{colog} (b + c) = \bar{2}.0222$	$\operatorname{colog} \sin C = \underline{0.2484}$
$\log \tan \frac{1}{2}(B - C) = \bar{1}.1712$	$\log a = 1.8481$
$\frac{1}{2}(B - C) = 8.44°$	$a = \boxed{70.48}$
$\frac{1}{2}(B + C) = 42.80°$	
$B = \boxed{51.24°}$	
$C = \boxed{34.36°}$	

Check by the formula $a \sin B = b \sin A$.

39. Case IV. Three Sides. As in Case III, triangles under Case IV may always be solved by the Law of Cosines. From $c^2 = a^2 + b^2 - 2\,ab \cos C$ we get

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab},$$

which is a form from which the angle may be computed.

Given: $a = 51.00$
$b = 65.00$
$c = 20.00$

$a^2 = 2601$	$2\,ab = 6630$
$b^2 = 4225$	$2\,bc = 2600$
$c^2 = 400.0$	$2\,ac = 2040$

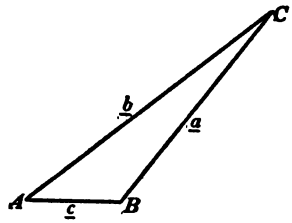


FIG. 44

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{2024}{2600} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2\,ac} = \frac{-1224}{2040}$$

$\log 2024 = 3.3062$	$\log 1224 = 3.0878$
$\log 2600 = \underline{3.4150}$	$\log 2040 = \underline{3.3096}$
$\log \cos A = \bar{1}.8912$	$\log (-\cos B) = \bar{1}.7782$

$$A = \boxed{38.88°} \qquad B = 180° - 53.13° = \boxed{126.87°}$$
$$C = 180° - (A + B) = \boxed{14.25°}$$

Check by means of the Law of Sines.

Or we may compute C directly by the method just used for A and B, and use the relation $A + B + C = 180°$ as a check.

If the numbers cannot be conveniently squared, the triangle may be solved by formulas better adapted to logarithmic computation. These formulas may be derived as follows:

The Half-angle Formulas. By dividing by one another formulas (6) and (7) of § 27, we find

$$(1) \qquad \tan \tfrac{1}{2} A = \sqrt{\frac{1 - \cos A}{1 + \cos A}},$$

where the positive sign is used before the radical because, A being an angle of a triangle and therefore less than 180°, $\frac{1}{2} A$ is necessarily acute, and therefore has a positive tangent.

We wish now to express the radical in terms of the sides of the triangle. For this purpose we use the Law of Cosines:

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

Hence
$$1 - \cos A = \frac{a^2 - (b - c)^2}{2\,bc} = \frac{(a + b - c)(a - b + c)}{2\,bc},$$
$$1 + \cos A = \frac{(b + c)^2 - a^2}{2\,bc} = \frac{(b + c + a)(b + c - a)}{2\,bc}.$$

These values substituted in (1) give the result we wish. The result may, however, be put in a more convenient form by means of the notation

$$(2) \qquad a + b + c = 2\,s.$$

We have then

$$a + b - c = 2(s - c),$$
$$a - b + c = 2(s - b),$$
$$b + c - a = 2(s - a).$$

Hence $$1-\cos A=\frac{2(s-b)(s-c)}{bc},$$

$$1+\cos A=\frac{2s(s-a)}{bc}.$$

These values substituted in (1) give the formula

$$\text{(3)}\qquad \tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

from which, by a mere change of letters, formulas for $\tan\frac{1}{2}B$, $\tan\frac{1}{2}C$ are found.

If instead of computing the third angle from the first two by means of the relation $A+B+C=180°$, we prefer to use this relation as a check and to compute all the angles directly, it will be advisable to make a slight change in formula (3). For this purpose, multiply numerator and denominator of the fraction under the radical in (3) by $s-a$. This gives

$$\tan\tfrac{1}{2}A=\frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Here, if we interchange a, b, c, the radical will not change. Hence we may compute this radical once for all and use it in all three formulas:

$$\text{(4)}\qquad r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

We then have

$$\text{(5)}\qquad \tan\tfrac{1}{2}A=\frac{r}{s-a},\quad \tan\tfrac{1}{2}B=\frac{r}{s-b},\quad \tan\tfrac{1}{2}C=\frac{r}{s-c}.$$

Meaning of r. The quantity r has a very simple geometric meaning which is of some importance, and may help to fix the formulas (5) in the memory.

Let O be the center of the circle inscribed in ABC and M, N, P its points of contact with the sides. Then, as we see from Figure 45,

$$\text{(6)}\qquad \tan\tfrac{1}{2}A=\frac{MO}{AM}.$$

Now by plane geometry

$$AM=AP,\quad BM=BN,\quad CP=CN.$$

Hence

Perimeter of triangle $= 2AM + 2BN + 2CN = 2AM + 2a.$

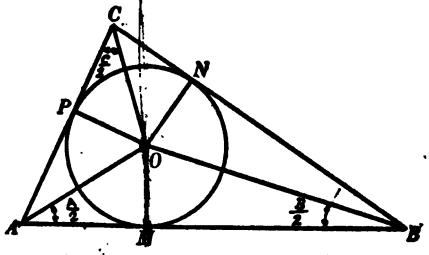


FIG. 45

But by (2) Perimeter of triangle $= 2s.$

Consequently $AM = s - a,$

so that we see from (6) that

$$MO = (s - a) \tan \tfrac{1}{2} A.$$

From (5), this is precisely the value of r. *Thus the quantity* r given by (4) and used in formulas (5) *is the radius of the circle inscribed in the triangle* ABC.

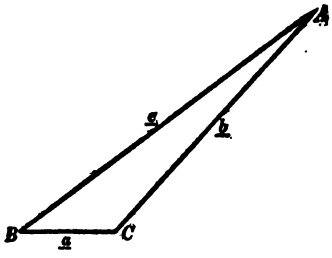


FIG. 46

Numerical Example.

Given:

$a =$	14.49
$b =$	55.43
$c =$	66.91
$2s =$	136.83
$s =$	68.42
$s - a =$	53.93
$s - b =$	12.99
$s - c =$	1.51

$\log (s - a) =$	1.7318
$\log (s - b) =$	1.1136
$\log (s - c) =$	0.1790
$\operatorname{colog} s =$	$\bar{2}.1648$
$\log r^2 =$	$\overline{1.1892}$

$\log r = 0.5946$	$\log r = 0.5946$	$\log r = 0.5946$
$\log (s - a) = \underline{1.7318}$	$\log (s - b) = \underline{1.1136}$	$\log (s - c) = \underline{0.1790}$
$\log \tan \frac{1}{2} A = \bar{2}.8628$	$\log \tan \frac{1}{2} B = \bar{1}.4810$	$\log \tan \frac{1}{2} C = 0.4156$
$\frac{1}{2} A = 4.17°$	$\frac{1}{2} B = 16.84°$	$\frac{1}{2} C = 68.99°$
$A = \boxed{8.34°}$	$B = \boxed{33.68°}$	$C = \boxed{137.98°}$

Check: $A + B + C = 8.34° + 33.68° + 137.98° = 180.00°.$

40. Area of a Triangle. If h is the length of the perpendicular dropped from B on the opposite side, we have by elementary geometry

(1) Area $ABC = \frac{1}{2} bh$.

If we substitute for h its value

$$h = c \sin A,$$

we find

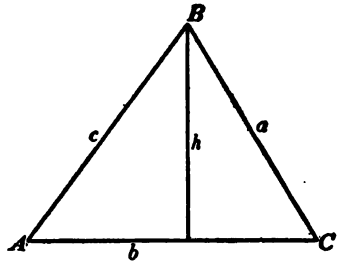


FIG. 47

(2) $$\text{Area } ABC = \tfrac{1}{2} bc \sin A,$$

or: *the area of a triangle is half the product of any two sides times the sine of the included angle.* This is by far the most important trigonometric formula for the area of a triangle.

Area in Terms of Sides. In Figure 45, § 39, ABC may be broken up into three triangles, OBC, OCA, OAB, each of which has r as an altitude, while the bases are a, b, c, respectively. The areas of these three triangles are therefore $\frac{1}{2} ar$, $\frac{1}{2} br$, $\frac{1}{2} cr$, and by adding these together we find

$$\text{Area } ABC = \tfrac{1}{2} (a + b + c)r,$$

or replacing $a + b + c$ by its value, $2s$,

(3) $$\text{Area } ABC = rs.$$

This is a convenient formula if we have already computed r. Otherwise, replacing r by its value, (4), § 39, we find

(4) $$\text{Area } ABC = \sqrt{s(s-a)(s-b)(s-c)}.$$

CHAPTER V

SPHERICAL RIGHT TRIANGLES

41. Geometrical Introduction. The student should provide himself with a sphere of some material on which marks can be made and erased, and a hemispherical cap which fits on the sphere. By using the cap as a ruler, great circles can be drawn in any position. The edge of this cap should be divided into degrees, so that, on any great circle, any desired arc can be marked off from a given point. The cap should also have a small hole at its center. If we rule a great circle and then, without moving the cap, mark a point on the sphere through the hole, this point is called the pole of the great circle, and bears to it exactly the relation either pole on the earth's surface bears to the equator.*

By means of this simple apparatus we can make readily many measurements and simple constructions on the spherical surface. For instance, we can measure, in degrees, the length of any arc of a great circle; or we can pass a great circle through any two points on the spherical surface. If we have two arcs of great circles intersecting at A, we can measure the angle between them† by placing the cap on the sphere so that the hole falls at A, and reading off on the edge the number of degrees between the two arcs (produced if necessary). On the other hand, if we have given a great circle AB, and wish to construct a second great circle through

* Apparatus of this character, but with various improvements of detail, has been used for many years in the instruction in astronomy given at Harvard University by Professor R. W. Willson.

† That is, the angle between their tangents at A. That this angle is given by the method here indicated will be evident even without knowledge of solid geometry. It is clear, for instance, that the angle between two meridians on the earth's surface is measured by the intercepted arc of the equator.

A making an angle of x degrees with it, we may place the cap so that the hole falls at A, and measure along its rim, starting from AB, an arc of $x°$, thus marking a point C on the great circle determined by the rim of the cap. Then draw the great circle AC.

The only other instrument we shall require is an ordinary compass for drawing small circles. The fixed point, P, of the compass is called the *pole* of the small circle, C, described by the moving point, and bears the same relation to it that a pole of the earth bears to a circle of latitude. If we pass various great circles through this pole, P, the distances measured along them from P to C are evidently all equal, and this distance, measured in degrees, is called the *polar distance* of the small circle. It is somewhat greater than the distance between the two points of the compass, and can easily be measured by means of the hemispherical cap.

Spherical Triangles. By a spherical triangle we understand the part of the spherical surface bounded by three arcs, AB, BC, CA, of great circles meeting at the three vertices, A, B, C. Not only will the angles of the spherical triangle be measured in degrees, but also its sides, these being arcs of great circles. Although it is easy to construct spherical triangles some of whose sides and angles are greater than 180°, we shall not consider any such triangles, but shall understand the term *spherical triangle* to mean one whose sides and angles are all less than 180°.

The angles of a spherical triangle we denote by A, B, C, the sides opposite by a, b, c, respectively, precisely as in the case of a plane triangle. When we are dealing with a right spherical triangle, we shall always assume $C = 90°$.

Polar Triangles. The side a of a spherical triangle ABC has two poles A' and A''. Let A' be that one of these two which lies on the same side of a as A. Similarly let B' be that pole of the circle b which lies on the same side of it as B;

and C' that pole of c which lies on the same side of it as C. Then the spherical triangle whose vertices are A', B', C' is called the polar triangle of ABC (Fig. 48). Its sides we denote by a', b', c'.

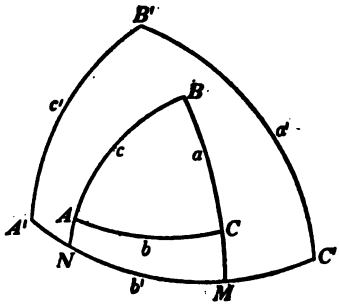


FIG. 48

By construction, B' is at a distance of 90° from every point on b, and C' is at this same distance from every point on c. Hence, since A lies on both b and c, B' and C' are both at a distance of 90° from A. Consequently, if we draw the great circle with pole at A, it passes through B', C' and hence coincides with a'. Moreover, since A and A' lie on the same side of a, it is easily seen that they also lie on the same side of a'. In the same way we see that B is that pole of b' which lies on the same side of it as B'; and that C is that pole of c' which lies on the same side of it as C'.

Thus we see that ABC is the polar triangle of $A'B'C'$, and the relation between the two triangles is a reciprocal one.

Let M and N be the points where the sides a and c, produced if necessary, meet b', or b' produced. Then, since A' and C' are at 90° distance from every point on a and c, respectively, $A'M = 90°$ and $CN = 90°$, and these two arcs are measured in opposite directions and overlap each other. Hence, adding them together,

$$180° = A'C' + NM.$$

But $A'C' = b'$, and NM is precisely the arc by means of which we measure the angle B. Thus the last written equation becomes

$$180° = b' + B,$$

and we may say that *any angle of a spherical triangle is the supplement of the corresponding side of the polar triangle.*

42. Graphical Solution of Right Spherical Triangles. Since $C = 90°$ is given, there remain five *parts* of the triangle, A, B, a, b, c. Moreover, there is no such equation between the angles as there is in the case of the plane triangle. In fact, it is true that for all spherical triangles *

$$A + B + C > 180°.$$

Consequently, we can give two parts of the right spherical triangle in the following essentially distinct ways :

I. b, c; II. a, b; III. b, A; IV. c, A; V. a, A; VI. A, B.

In the graphical solutions of these six cases which we now give we assume that the given sides and angles are all less than 180°.

I. Given b and c. Construct two great circles making a right angle with one another at C. Along one of these circles lay off the distance $CA = b$. From A as pole construct a circle (in general a small circle) with polar distance c. Note the point B where this meets the other circle. ABC is the desired triangle, and its unknown parts, a, A, B, can be measured. If $b < 90°$, there is here no solution when $c \leqq b$ or when $c \geqq 180° - b$, since in either of these cases the small circle described about A as pole will not meet the second great circle through C. Similarly, when $b > 90°$, there is no solution when $c \geqq b$ or $c \leqq 180° - b$. There are otherwise ap-

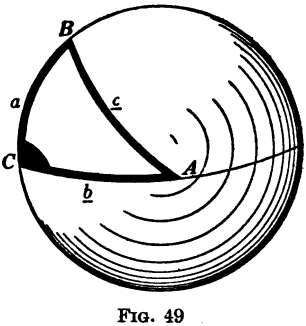


FIG. 49

* To see this, notice first that in any spherical triangle the sum of the sides is less than a great circle. This is sufficiently obvious without demonstration, since the triangle lies wholly on a hemisphere, so that its perimeter must be shorter than the perimeter of the hemisphere. It follows then that, in the polar triangle,

$$a' + b' + c' < 360°,$$

or $$(180° - A) + (180° - B) + (180° - C) < 360°.$$

Hence $$180° < A + B + C.$$

parently two solutions, since the small circle will then meet CB in two points. The two triangles thus formed are, however, entirely symmetrical, lying in just the same way on opposite sides of AC. They have, then, exactly the same parts and do not give different solutions of the problem. * Consequently, there is just one solution when $b < 90°$ and $b < c < 180° - b$ and also when $b > 90°$ and $180° - b < c < b$. Otherwise there is no solution except in the one case $b = c = 90°$, when there are an infinite number of solutions.

II. Given a and b. Construct two great circles making a right angle at C. From C along these circles lay off CA and CB equal to b and a, respectively. Join A and B by a great circle. The triangle ABC is the desired triangle. It is clear from this construction that this problem has one and only one solution.

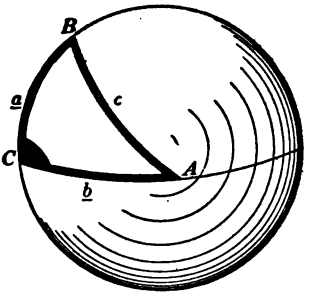


FIG. 50

III. Given b and A. Here again an obvious graphical construction leads to a solution; and we see that there is always just one solution in this case.

IV. Given c and A. Construct two great circles intersecting at A and making there the desired angle A with each other. Construct the pole P of the first of these circles. On the second lay off $AB = c$. Construct the great circle BP, and let C be the point where it

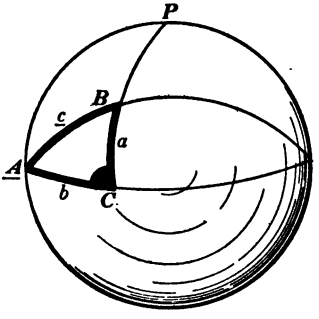


FIG. 51

* It should be noted that two symmetrical spherical triangles are not usually equal in the sense of being superposable.

meets the first circle. Then ABC is the desired triangle, since C is obviously a right angle. Here we see that there is just one solution, except that, if $A = 90°$, there is either no solution, or, when we also have $c = 90°$, an infinite number of solutions.

V. Given a and A. Construct two great circles intersecting at A and A', and making the desired angle with each other, and let P be the pole of the first of them. Let us first assume that A and a are both less than 90°. With P as pole and the polar distance $90° - a$ construct a small circle meeting the second great circle in B. Let the great circle PB meet the first great circle in C. Then ABC is the desired triangle, but $A'BC$ also satisfies the conditions of the problem, and we have here an ambiguous case. It might seem that a further ambiguity would be introduced by the fact that the small circle meets ABA' at a second point B', thus giving two more triangles, $AB'C'$ and $A'B'C'$, but an examination of the figure will show that ABC and $A'B'C'$ have exactly the same parts, being symmetrical triangles, and that the same is true of $A'BC$ and $AB'C'$. Thus we get only two solutions in all. If the two points B and B' coincide, that is, if $A = a$, we have only one solution; and if the small circle is too small to meet ABA', we have no solution. On the other hand, if, A being still acute, $a > 90°$, there is clearly no solution.

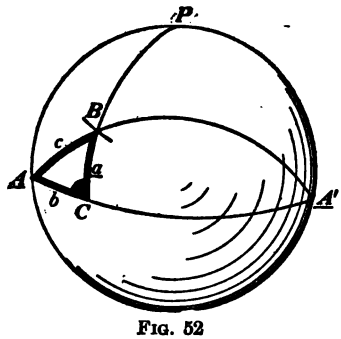


FIG. 52

If we next turn to the case in which A and a are both greater than 90°, a precisely similar construction is possible in which the polar distance of the small circle is $a - 90°$; while if A is obtuse but $a < 90°$, there is clearly no solution.

The whole situation may now be readily summarized as follows:

Two solutions if $90° > A > a$ or $90° < A < a$;

One solution if $A = a \neq 90°$;

Infinitely many solutions if $A = a = 90°$;

No solution in all other cases.

VI. Given A and B. Since here all three angles are given, in the polar triangle, $A'B'C'$, all three sides are known; namely,

$$a' = 180° - A, \quad b' = 180° - B, \quad c' = 180° - 90° = 90°.$$

We first construct the polar triangle by laying off on a great circle an arc $A'B' = 90°$. Then with A' and B' as poles, construct small circles with polar distances $180° - A$ and $180° - B$, respectively. Let C' be a point where these circles meet. (The second point, C'', where they meet gives a symmetrical triangle and need not be considered.) $A'B'C'$ is the polar triangle, and by measuring its angles and taking their supplements we obtain the sides of the desired triangle, ABC; or, if we prefer, the desired triangle can be constructed as the polar triangle of $A'B'C'$. If, however, the two small circles in our construction do not meet, or only just touch without intersecting, the problem has no solution, and this occurs when

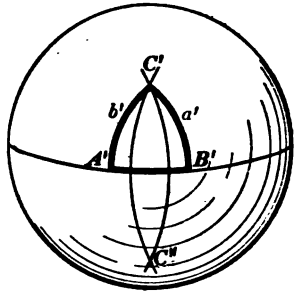


FIG. 53

$$a' + b' \leqq c', \qquad \text{that is } 360° - A - B \leqq 90°,$$

$$\text{or } a' + b' + c' \geqq 360°, \qquad \text{that is } 360° - A - B + 90° \geqq 360°,$$

$$\text{or } a' - b' \geqq c', \qquad \text{that is } B - A \geqq 90°,$$

$$\text{or } b' - a' \geqq c', \qquad \text{that is } A - B \geqq 90°.$$

Hence, if the two given angles, A and B, are such that their sum is less than 270° but greater than 90°, while their difference is less than 90°, there is just one solution. When either of these conditions is violated, there is no solution.

43. The Trigonometric Formulas. Let ABC be any right spherical triangle with right angle at C, and O the center of the sphere. Draw the radii OA, OB, OC. For convenience we shall think of the plane OBC as vertical, the plane OAC as horizontal, a position in which we can always place our figure, since the angle C is a right angle. Pass a plane, BDE, through B perpendicular to OA, so that OE is perpendicular to both EB and ED. The angle BED is then equal to the angle A of the spherical triangle, since this last angle is, by definition, equal to the angle between the tangents to AB and AC at A, and these tangents are parallel to EB and ED, respectively.

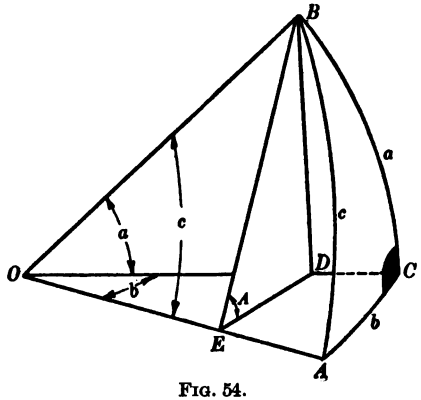


FIG. 54.

The plane BDE is clearly a vertical plane, since it is perpendicular to the horizontal line OA. Consequently, the triangles

BDO and BDE are right triangles, $D = 90°$.

Also, from what was said above,

BEO and DEO are right triangles, $E = 90°$.

Finally, since the angles BOC, BOA, AOC are measured respectively by the arcs BC, BA, AC, we see that these angles have the values marked in Figure 54.

We will call the radius of the sphere R.

Now OE can be regarded as the projection both of $OB = R$, and of OD. Hence

$$R \cos c = OE,$$

and

$$OE = \overline{OD} \cos b.$$

Consequently

$$R \cos c = \overline{OD} \cos b.$$

But OD is the projection of $OB = R$, and thus

$$OD = R \cos a.$$

Hence, substituting and canceling out R, we find

$$\cos c = \cos a \cos b.$$

This may be called the Pythagorean Theorem for spherical geometry, since it expresses the hypotenuse of any right spherical triangle in terms of the sides.

Again, from Figure 54 we have

$$\sin A = \frac{DB}{EB},$$

but

$$DB = R \sin a,\ EB = R \sin c.$$

Hence

$$\sin A = \frac{\sin a}{\sin c}.$$

This may be called the *sine-formula.* It tells us that *the sine of an angle of a spherical right triangle is equal to the sine of the opposite side divided by the sine of the hypotenuse.* It may be easily remembered from its similarity to the corresponding formula for the sine of an angle of a plane right triangle.

A *tangent formula* may be found as follows:

$$\tan A = \frac{DB}{ED} = \frac{R \sin a}{OD \sin b} = \frac{R \sin a}{R \cos a \sin b} = \frac{\tan a}{\sin b}.$$

That is, *the tangent of an angle of a spherical right triangle is equal to the tangent of the opposite side divided by the sine of the adjacent side.* Here again the analogy to plane triangles is evident.

The three principles so far deduced yield at once formulas (1)–(5) below, the sine-formula being applied first to A and

then to B in order to give (2) and (3), and the tangent-formula in a similar way to give (4) and (5).

The other five formulas, (6)–(10), may be deduced from the first five without reference to the figure. For instance, dividing (2) by (4) and replacing $\tan a$ by $\frac{\sin a}{\cos a}$ gives

$$\cos a = \frac{\sin c}{\sin b} \cos A.$$

If here we replace $\cos a$ by its value from (1), we find readily

$$\cos A = \frac{\tan b}{\tan c},$$

which may be called the *cosine-formula* and is substantially formula (6) below. In words it says *the cosine of an angle of a right spherical triangle is equal to the tangent of the adjacent side divided by the tangent of the hypotenuse.*

Applied to the angle B, this principle gives formula (7).

The student should deduce formulas (8), (9), (10) from formulas (1)–(5).

(1) $\cos c = \cos a \cos b$
(2) $\sin a = \sin c \sin A$
(3) $\sin b = \sin c \sin B$
(4) $\tan a = \sin b \tan A$
(5) $\tan b = \sin a \tan B$
(6) $\tan b = \tan c \cos A$
(7) $\tan a = \tan c \cos B$
(8) $\cos A = \cos a \sin B$
(9) $\cos B = \cos b \sin A$
(10) $\cos c = \operatorname{ctn} A \operatorname{ctn} B$

Formulas (1)–(5) have been derived by using a figure in which all the parts, except C, are acute. By considering other figures, like Figure 55, the student should satisfy himself that these formulas, and hence also (6)–(10) which were derived from them, are true in all cases.

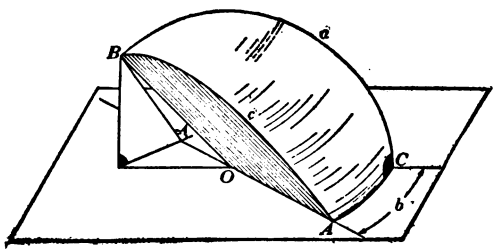


FIG. 55

There are certain consequences of formulas (1)–(5) which we now note, since we shall find them useful in § 45 in the actual numerical solutions of triangles. In each case the student should deduce the statement from the formula in question.

From (1):

I. *If* $c < 90°$, *a and b are either both less or both greater than* $90°$; *if* $c > 90°$, *one is less and the other greater than* $90°$.

From (4) and (5):

II. *A and a are either both less or both greater than* $90°$, *and the same is true of B and b.*

44. Spherical Right Triangles which are almost Plane. If the lengths of the sides of a spherical triangle are very small in comparison to the radius of the sphere, the portion of the spherical surface where the triangle lies is almost a plane. This is, for instance, the case for a triangle of moderate dimensions (with sides not more than forty or fifty miles long) on the surface of the earth. In this case the formulas of spherical trigonometry should reduce substantially to those of plane trigonometry. That they actually do so may be seen as follows:

The sides of such a spherical triangle as we are now considering subtend very small angles at the center of the sphere, so that the cosines of these angles are very nearly equal to 1, while their sines and tangents are very nearly what we get by dividing the lengths of the sides by the radius R of the sphere, the lengths of the sides being, of course, now measured not in degrees, but in terms of the same unit of length as is used to measure the radius of the sphere. In the sine, tangent, and cosine formulas mentioned in § 43 (and hence also in formulas (2)–(7), which are equivalent to them) we may therefore with very slight error replace the sines and tangents of the sides of the spherical triangle by the lengths of these sides. When this is done, we obtain the ordinary formulas for the plane right triangle.

In order to reduce formula (1), § 43, let us replace the cosines of the sides by their values in terms of the sines. On squaring, (1) takes the form

$$1 - \sin^2 c = (1 - \sin^2 a)(1 - \sin^2 b),$$

or

$$\sin^2 c = \sin^2 a + \sin^2 b - \sin^2 a \sin^2 b.$$

If now we use a, b, c to denote the lengths of the sides in linear measure (not in degrees), we may, with only very slight error, write the last equation in the form

$$\frac{c^2}{R^2} = \frac{a^2}{R^2} + \frac{b^2}{R^2} - \frac{a^2}{R^2}\frac{b^2}{R^2},$$

or
$$c^2 = a^2 + b^2 - a^2\frac{b^2}{R^2}.$$

Since R is very large in comparison to a and b, the last term may be neglected, and we see that, in the case of the triangles we are considering in this section, the Pythagorean proposition of spherical trigonometry (formula (1), § 43) reduces very approximately to the Pythagorean proposition for plane geometry.

From what has been said in this section it is clear why in most problems in surveying, plane and not spherical trigonometry is used, since the former is sufficiently accurate and simpler. The same remark holds for problems in navigation where the distances covered are not too great. This method of *plane sailing*, as it is called, is justified over much larger areas than would be the use of the formulas of plane trigonometry in surveying problems, since the degree of accuracy possible or desirable is so much less in problems in navigation.

45. Napier's Rules, and the Trigonometric Solution of Spherical Right Triangles. Formulas (1)–(10) of § 43 may be collected into very compact and convenient form by means of a rule formulated by John Napier, the inventor of logarithms, early in the seventeenth century. Let us mark, as in Figure 56, the two sides, a and b, and the complements of the other three parts. The right angle, C, we do not mark at all. Thus we have five "circular parts" arranged in succession around the triangle. When we start from any part, there are two *adjacent* parts, and the other two parts are called *opposite* parts. With the understanding that the parts spoken of are those marked in the figure, some of which are the complements of the true parts of the triangle, we may state

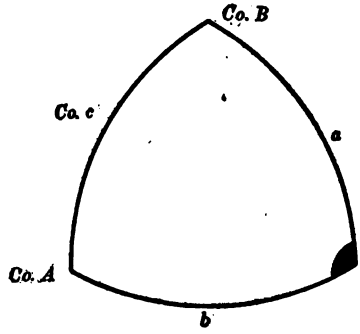


FIG. 56

NAPIER'S RULES:

1. *The sine of any part equals the product of the tangents of the adjacent parts.* (tan. ad., note the two *a*'s.)

2. *The sine of any part equals the product of the cosines of the opposite parts.* (cos. op., note the two *o*'s.)

The student should prove that these rules are correct by applying them in succession to all five parts of the figure. He will thus get ten formulas, which, when reduced to their simplest forms, will turn out to be equivalent to the ten of § 43.

In solving numerical triangles, a part and its two adjacent parts (or else a part and its two opposite parts) should be selected in such a way that two of these three parts are known. Napier's Rule will then give an equation for determining the third part.

Example 1.

Given: $a = 61.00°$

$B = 123.67°$

This comes under Case III, § 42, and hence there is just one solution.

To find A:	To find c:	To find b:
$\sin \text{co}A$	$\sin \text{co}B = \tan a \tan \text{co}\, c$	$\sin a = \tan \text{co}B \tan b$
$= \cos \text{co}B \cos a$	or* $-\text{ctn}\, c = \dfrac{-\cos B}{\tan a}$	$-\tan b = \dfrac{\sin a}{-\text{ctn}\, B}$
$\cos A = \sin B \cos a$		$= \sin a(-\tan B)$
		$\log \sin a = \bar{1}.9418$
$\log \sin B = \bar{1}.9203$	$\log(-\cos B) = \bar{1}.7438$	$\log(-\tan B) = 0.1764$
$\log \cos a = \bar{1}.6856$	$\log \tan a = 0.2562$	$\log(-\tan b) = \bar{0}.1182$
$\log \cos A = \bar{1}.6059$	$\log(-\text{ctn}\, c) = \bar{1}.4876$	$180° - b = 52.70°$
$A = \boxed{66.20°}$	$180° - c = 72.91°$	$b = 180° - 52.70°$
	$c = 180° - 72.91°$	$b = \boxed{127.30°}$
	$c = \boxed{107.09°}$	

Check: $\cos A = \tan b \, \text{ctn}\, c$.

* Since $\cos B$ and hence $\text{ctn}\, c$ are negative, we have changed the sign on both sides of this equation so as to be able to deal with nothing but positive quantities. This is essential to logarithmic work, since negative quantities have no logarithms.

Example 2. Ambiguous Case. (Case V, § 42).

Given: $B = 80.00°$
$b = 67.67°$

To find c:	To find A:	To find a:
$\sin b = \cos \text{co}B \cos \text{co}\, c$	$\sin \text{co}B = \cos \text{co}A \cos b$	$\sin a = \tan \text{co}B \tan b$
$\sin c = \dfrac{\sin b}{\sin B}$	$\sin A = \dfrac{\cos B}{\cos b}$	$\sin a = \text{ctn}\, B \tan b$
$\log \sin b = \bar{1}.9661$	$\log \cos B = \bar{1}.2397$	$\log \text{ctn}\, B = \bar{1}.2463$
$\log \sin B = \bar{1}.9934$	$\log \cos b = \bar{1}.5797$	$\log \tan b = \underline{0.3865}$
$\log \sin c = \bar{1}.9727$	$\log \sin A = \bar{1}.6600$	$\log \sin a = \bar{1}.6328$
$c_1 = \boxed{69.90°}$	$A_1 = \boxed{27.20°}$	$a_1 = \boxed{25.42°}$
$c_2 = \boxed{110.10°}$	$A_2 = \boxed{152.80°}$	$a_2 = \boxed{154.58°}$

Check: $\sin a = \sin A \sin c$.

Since all these unknown parts, c, A, a, are determined by their sines, we are led to an acute and an obtuse angle for each of them. We know by § 42 that the triangle has just two solutions. A reference to Rules I and II at the end of § 43 will show how the six values of c, A, and a which we have found must be grouped together to give these solutions. It happens, in this case, that the three acute values give one triangle, and the three obtuse angles another: but this is an accident.

Example 3. Given a and c, to find A. By applying the second of Napier's Rules to a, we find as the formula to be used

$$\sin A = \frac{\sin a}{\sin c}.$$

Thus A is determined by its sine, and we appear to have a choice between an acute and an obtuse value. The case, however, is not really an ambiguous one, and by using Rule II, § 43, we easily determine, in any numerical case, which value of A should be taken.

In the same way it will be found that all apparent, but not real, ambiguities which present themselves in the trigonometric solution of triangles may be removed by a reference to the two rules of § 43.

46. Special Oblique Triangles Solved by Means of Right Triangles. While the general method for solving oblique spherical triangles will be explained in the next chapter, there are certain simple cases in which the solution can be more easily effected by means of the principles already explained.

Quadrantal Triangles. If one of the sides of a spherical triangle is 90° (a quadrant), the triangle is called a quadrantal triangle. Since each side of any spherical triangle is the supplement of the corresponding angle of its polar triangle, the polar triangle in this case will be a right triangle. We can solve the polar triangle, and then find the parts of the original triangle as the supplements of those computed.

Isosceles Triangles. An isosceles spherical triangle is one in which two sides are equal. It is obviously true, as in the case of plane triangles, that the opposite angles are also equal, and, conversely, that if two angles of a spherical triangle are equal, the triangle is isosceles. Moreover, the great circle connecting the vertex of an isosceles spherical triangle with the middle point of the base will be perpendicular to the base, and will bisect the angle opposite. Thus this perpendicular divides the triangle into two right triangles symmetrical to each other and having the same values for their parts. The isosceles triangle has four distinct parts: the base, one side, one base angle, and the vertical angle. When any two of these are given, one of the right triangles just mentioned can be solved and thus the other parts found.

Other Oblique Triangles. In other cases also we may pass a great circle through one vertex of the triangle perpendicular to the opposite side, thus forming two right triangles. This can frequently be done in such a way that the given parts of the original triangle yield us the parts necessary in order to make the solution of the right triangles possible. When the parts of the right triangles are known, the parts of the original triangle can be found easily. For instance, if two sides and the included angle are given, we can solve by substantially the same method as that used for the corresponding case of plane triangles in § 38.

CHAPTER VI

THE OBLIQUE SPHERICAL TRIANGLE

47. The Law of Sines. We consider in this chapter a spherical triangle ABC with sides a, b, c, all of whose sides and angles are less than 180°.

Through a vertex, C, and the poles of the opposite side draw a great circle cutting the side AB, produced if necessary, in D. This great circle is perpendicular to AB, and we denote its arc CD by h. Since we may choose as D either point where the great circle cuts AB, and since one or both of these points will lie on AB produced, a variety of figures are possible, of which Figure 57 is one. The changes necessary in the reasoning in this section and the next, when we have to deal with one of the other figures, are very slight, and we leave it to the reader to satisfy himself that they make no difference to the final result.

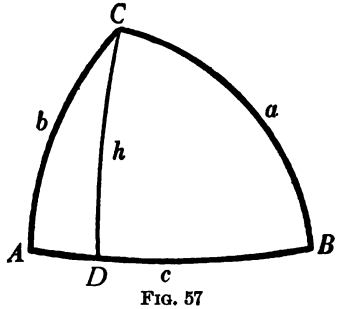


FIG. 57

By applying the sine formula for right spherical triangles (§ 43) to the angles A and B in the two right triangles ADC and BDC, we find

$$\sin A = \frac{\sin h}{\sin b},$$

$$\sin B = \frac{\sin h}{\sin a}.$$

Hence, by division,

$$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}, \tag{1}$$

or, in words, *the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.*

Instead of formula (1) and others similar to it obtained by changing the letters, we may write more symmetrically

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{2}$$

It should be noticed how closely analogous both the final results and the method of proof are to the corresponding case in plane trigonometry (§ 32).

48. The Law of Cosines. Using the same figure and notation as above, we have, by the Pythagorean Theorem for spherical geometry (formula (1), § 43),

$$\cos a = \cos h \cos \overset{\frown}{DB},$$

$$\cos b = \cos h \cos \overset{\frown}{AD}.$$

Hence, by division,

$$\frac{\cos a}{\cos b} = \frac{\cos \overset{\frown}{DB}}{\cos \overset{\frown}{AD}} = \frac{\cos (c - \overset{\frown}{AD})}{\cos \overset{\frown}{AD}} = \cos c + \sin c \tan \overset{\frown}{AD}.$$

But by the cosine formula for right spherical triangles (formula (6), § 43),

$$\tan \overset{\frown}{AD} = \tan b \cos A.$$

Substituting this above and clearing of fractions, we find

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \tag{1}$$

The cosine of any side of a spherical triangle is equal to the product of the cosines of the other two sides plus the product of their sines into the cosine of the included angle.

As in the case of plane triangles, the Law of Cosines may be regarded as a generalization of the Pythagorean Theorem.

49. Formulas for the Half-Angles. By methods very similar to those used in § 39, we will now deduce formulas very similar to the formulas of that section.

From the Law of Cosines we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Hence,

$$1 - \cos A = \frac{\cos (b - c) - \cos a}{\sin b \sin c},$$

$$1 + \cos A = \frac{\cos a - \cos (b + c)}{\sin b \sin c}.$$

Consequently,

$$\tan \tfrac{1}{2} A = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \sqrt{\frac{\cos (b - c) - \cos a}{\cos a - \cos (b + c)}}.$$

By means of the formula

$$\cos x - \cos y = -2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y)$$

(see (2), § 28), we may write

$$\cos (b - c) - \cos a = -2 \sin \tfrac{1}{2}(a + b - c) \sin \tfrac{1}{2}(b - c - a),$$
$$\cos a - \cos (b + c) = -2 \sin \tfrac{1}{2}(a + b + c) \sin \tfrac{1}{2}(a - b - c).$$

Let us then write

$$2s = a + b + c, \tag{1}$$

so that

$$2(s - a) = -a + b + c, \quad 2(s - b) = a - b + c, \quad 2(s - c) = a + b - c.$$

Our formula then becomes

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}. \tag{2}$$

By letting

$$p = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}, \tag{3}$$

we get the first of the following formulas, from which the other two are obtained by an interchange of letters:

$$\tan \tfrac{1}{2} A = \frac{p}{\sin (s-a)},$$

$$\tan \tfrac{1}{2} B = \frac{p}{\sin (s-b)}, \tag{4}$$

$$\tan \tfrac{1}{2} C = \frac{p}{\sin (s-c)}.$$

50. The Solution of Spherical Triangles. Of the six parts of a spherical triangle, three can be given in six essentially different ways.

I. Three sides.
II. Two sides and the included angle.
III. Two sides and an angle opposite one of them.
IV. Two angles and a side opposite one of them.
V. Two angles and the included side.
VI. Three angles.

We will consider these cases one by one.

Case I is analogous to Case IV for plane triangles (§ 39) and may be solved by substantially the same method; namely, by means of formulas (4), § 49. The check, $A + B + C = 180°$, which we had for plane triangles, is no longer available, but we may check any two of the computed parts by a single application of the Law of Sines.

Geometrically it is clear that there will be no solution if any one of the following inequalities is fulfilled: *

$$a + b \leqq c, \quad b + c \leqq a, \quad a + c \leqq b, \quad a + b + c \geqq 360°.$$

In all other cases a simple and obvious geometric construction shows that there is just one solution. In the trigonometric solution just given there is also no ambiguity since the half-angles are given by their tangents.

* For this last inequality see the first footnote to § 42.

Case VI may be treated by noticing that when the angles are given, the sides of the polar triangle may be at once computed as their supplements. The polar triangle thus comes under Case I and may be solved by the method indicated above. The sides of the original triangle are then computed as the supplements of the angles of the polar triangle.

It is clear that there is never more than one solution, since the polar triangle comes under Case I, and hence has only one solution. We leave it for the reader to specify the cases in which there is no solution.

Case II. The Law of Cosines gives us immediately the side opposite the given angle. The three sides being now known, the method of Case I may be applied for obtaining the remaining angles. Various checks will at once suggest themselves.

It is clear from an obvious geometrical construction that there is, in this case, always just one solution.

Case V may be treated by using the polar triangle and thus reducing the problem to Case II.

Here again there is always just one solution.

Case III. Denote the given sides by a and b, the given angle by A. The angle B may be computed by means of the Law of Sines. When this has been done, c and C are most readily computed by drawing through C a great circle perpendicular to c, thus forming two right triangles (Fig. 57, § 47). Solving these triangles by Napier's Rules, we find the arcs AD and DB, whose sum (or difference) equals c, and also the angles at C whose sum (or difference) equals C.

The form of the Law of Sines involving B and C may be used as a check.

When B has once been computed, there is no ambiguity in the method just given for obtaining c and C. Since B is to be computed from its sine, we are left in doubt whether the acute or obtuse value for B should be taken. This am-

biguity in the method of solution corresponds frequently, as we shall see in the next section, to a real ambiguity in the problem.

Case IV may be reduced to Case III by considering the polar triangle. There will be the same ambiguity in the solution here as in Case III.

51. Ambiguous Cases. We have seen that Cases III and IV, § 50, are the only ones in which ambiguities can arise. We begin with Case III, in which A, a, b are given.

Suppose, first, that $A < 90°$.

Construct two half great circles running from A to A' and making the desired angle, A, with each other. Lay off the arc $AC = b$ on the first of them, and with C as pole and polar distance a describe a small circle cutting the second great circle in B_1 and B_2. The triangles ACB_1 and ACB_2 give the two solutions of the problem.

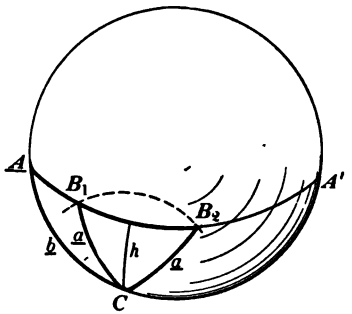


FIG. 58

It may, however, happen that the small circle described about C is too small to meet the second great circle AA', in which case there is no solution. It may also happen that one or both of the points B_1, B_2 lie not on the semi-circle AA' but on this semi-circle produced, and here again there will be only one solution or no solution. It is easy now to see that we may make the following classification:

$A < 90°$.

1) $a \geqq b$ and also $a \geqq 180° - b$. No solution.
2) $a < b$ and also $a < 180° - b$. Doubtful case (0, 1, or 2 solutions).
3) All other cases (a between b and $180° - b$ or equal to the smaller of them). One solution.

By the sine formula for right spherical triangles (formula (2), § 43) it is clear that the shortest (perpendicular) distance, h, from C to the arc AA' is given by the formula

$$\sin h = \sin b \sin A.$$

Consequently, in what we have called the doubtful case 2), we shall have 0, 1, or 2 solutions according as h is greater than, equal to, or less than a, that is (since a and h are necessarily both less than 90°), according as $\sin b \sin A$ is greater than, equal to, or less than $\sin a$. Since we propose to determine B by the Law of Sines,

$$\sin B = \frac{\sin b \sin A}{\sin a},$$

we shall therefore have 0, 1, 2 solutions, according as the computed value for $\sin B$ is greater than, equal to, or less than 1.

Consequently, in what we have called the doubtful case, the rule is *to carry through the trigonometric solution for the angle B and to retain all the answers we get.*

On the other hand, in case 3) the trigonometric solution gives us no clue as to how many solutions there are. There will always *appear* to be two solutions, but one of them must be discarded. It is easy to see from the figure that B must be taken as acute or obtuse, according as b is less than or greater than 90°.

In a similar way we can discuss the cases in which A is obtuse, with the following results:

$A > 90°$:

4) $a \leqq b$ and also $a \leqq 180° - b$. No solution.
5) $a > b$ and also $a > 180° - b$. Doubtful case (0, 1, or 2 solutions).
6) All other cases (a between b and $180° - b$ or equal to the larger of them). One solution.

Here, too, in what we have called the doubtful case, 5), we retain all solutions which the trigonometric work gives;

while in case 6) we discard one value of B, retaining the acute value if $b < 90°$, the obtuse if $b > 90°$.

The case $A = 90°$ need not be discussed here, since it is precisely Case I of § 42, except that the given angle A was there called C.

We leave it for the student to discuss Case IV in a similar manner. Such a discussion will be unnecessary if we reduce Case IV to Case III by use of the polar triangle.

52. The Law of Tangents. From the Law of Sines we deduce, by precisely the method of § 34, the formula

$$\frac{\sin a - \sin b}{\sin a + \sin b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

The reduction used in § 34 for the second member may now be applied to both members, and we find

$$\frac{\tan \tfrac{1}{2}(a-b)}{\tan \tfrac{1}{2}(a+b)} = \frac{\tan \tfrac{1}{2}(A-B)}{\tan \tfrac{1}{2}(A+B)}. \tag{1}$$

This is the Law of Tangents in spherical trigonometry.

53. Dual Formulas. By applying the Law of Cosines to the polar triangle $A'B'C'$, with sides a', b', c', we get

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

When we make use of the relations

$$a' = 180° - A, \quad b' = 180° - B, \quad c' = 180° - C, \quad A' = 180° - a,$$

this formula reduces to

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \tag{1}$$

The method just used is a very important one in spherical trigonometry. Whenever a formula has been obtained valid for all spherical triangles, we may apply it to the polar triangle of the given triangle and thus deduce a new formula, called the *dual* of the first. Thus formula (1) of this section is the dual of formula (1), § 48, and, conversely, it may be shown that the formula of § 48 is the dual of the formula of this section.

We may, of course, obtain other formulas by changing the letters in (1), which will be the duals of the other forms in which the Law of

Cosines may be written; and we may enunciate the contents of these formulas by saying

The cosine of any angle of a spherical triangle is equal to the negative of the product of the cosines of the other two angles plus the product of their sines into the cosine of the included side.

This theorem may be called the dual of the Law of Cosines.

Occasionally it will happen that when we form the dual of a formula or a theorem we get precisely the same formula or theorem over again. In this case the formula or theorem may be said to be self-dual. Examples of this are the Law of Sines and the Law of Tangents.

In general, however, the dual formulas will be different. As the formulas dual to those of § 49 we readily find.

(2)
$$\operatorname{ctn} \tfrac{1}{2} a = \frac{P}{\cos (S - A)}, \quad \operatorname{ctn} \tfrac{1}{2} b = \frac{P}{\cos (S - B)}, \quad \operatorname{ctn} \tfrac{1}{2} c = \frac{P}{\cos (S - C)},$$

where

(3)
$$2S = A + B + C,$$

(4)
$$P = \sqrt{\frac{\cos (S - A) \cos (S - B) \cos (S - C)}{-\cos S}}.$$

54. Napier's Analogies. By division, we obtain from (4), § 49

$$\frac{\tan \tfrac{1}{2} A}{\tan \tfrac{1}{2} B} = \frac{\sin (s - b)}{\sin (s - a)}.$$

Let us first add and then subtract 1 from both sides of this equation This gives

$$\frac{\tan \tfrac{1}{2} A \pm \tan \tfrac{1}{2} B}{\tan \tfrac{1}{2} B} = \frac{\sin (s - b) \pm \sin (s - a)}{\sin (s - a)}.$$

Dividing one of these equations by the other, we get

(1)
$$\frac{\tan \tfrac{1}{2} A - \tan \tfrac{1}{2} B}{\tan \tfrac{1}{2} A + \tan \tfrac{1}{2} B} = \frac{\sin (s - b) - \sin (s - a)}{\sin (s - b) + \sin (s - a)}.$$

The first member of this equation reduces, when we express the tangents in terms of sines and cosines, to

$$\frac{\sin \tfrac{1}{2} A \cos \tfrac{1}{2} B - \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B}{\sin \tfrac{1}{2} A \cos \tfrac{1}{2} B + \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B} = \frac{\sin \tfrac{1}{2} (A - B)}{\sin \tfrac{1}{2} (A + B)}.$$

To reduce the second member of (1), we use the formula

$$\frac{\sin x - \sin y}{\sin x + \sin y} = \frac{\tan \tfrac{1}{2} (x - y)}{\tan \tfrac{1}{2} (x + y)}$$

obtained by division from formulas (2), § 28. By means of this formula, the second member of (1) becomes

$$\frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}(2s-a-b)} = \frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}c}.$$

By means of these reductions, formula (1) now becomes

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c. \tag{2}$$

By dividing this by formula (1), § 52, we get

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c. \tag{3}$$

The duals of (2) and (3) are

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} ctn \tfrac{1}{2}C. \tag{4}$$

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} ctn \tfrac{1}{2}C. \tag{5}$$

These four formulas, (2)–(5), are called Napier's Analogies,* having been discovered by Napier, the inventor of logarithms.

Two other sets of four formulas each may, of course, be obtained from them by a mere interchange of letters.

55. Improved Solutions of Triangles. By means of the formulas obtained in the preceding sections, particularly Napier's Analogies, the trigonometric solutions of oblique spherical triangles indicated in § 50 can be somewhat improved. We refer to the cases of § 50 by number.

In Case I no improvement is possible, except that all three parts may be checked by the use of any one of Napier's Analogies.

Case VI, instead of being reduced to Case I by the use of the polar triangle, may now be treated directly by means of formulas (2), § 53. It should be noticed that these formulas are the duals of those we use in treating Case I.

Case II may now be treated by means of Napier's Analogies, which are well adapted to logarithmic work, instead of the Law of Cosines. If a, b, C are given, we first compute the values of $\frac{1}{2}(A-B)$ and $\frac{1}{2}(A+B)$ by means of formulas (4), (5), § 54. By addition and subtraction we get from these the values of A and B. The side c may then be computed by means of either (2) or (3), § 54.

* The word *analogy* is here used in an obsolete sense meaning *proportion*.

If only c is desired, the method of § 50 is preferable.

Case V, instead of being reduced to Case II by the use of the polar triangle, may be treated directly by means of the formulas dual to those we have just employed in the treatment of Case II. That is, if A, B, c are given, we first compute a and b by means of (2), (3), § 54; then C by means of (4) or (5).

In Case III, if we have given a, b, A, we begin, as in § 50, by computing B by means of the Law of Sines, using the discussion of § 51 to decide which solutions should be retained. The remaining parts, c and C, may be now computed without further ambiguity by means of formula (2) or (3) and formula (4) or (5) of § 54.

Case IV, instead of being reduced to Case III by the use of the polar triangle, may be treated directly by means of the formulas dual to those we have just employed in the treatment of Case III. That is, if A, B, a are given, we first compute b by means of the Law of Sines; then we compute C by means of (4) or (5), § 54, and c by means of (2) or (3).

EXERCISES

CHAPTER I

1. Definitions of the Trigonometric Functions.

1. Find the values of all the functions of the angle A in the triangle ABC whose sides, a, b, c, are respectively

(a) 3, 4, 5 (c) 9, 40, 41 (e) 1, 1, $\sqrt{2}$

(b) 5, 12, 13 (d) 1, $\sqrt{3}$, 2 (f) 11, 60, (?)

2. Find the values of the functions of B when

(a) $a = x$, $b = y$, $c = \sqrt{x^2 + y^2}$

(b) $a = 2\sqrt{mn}$, $b = m - n$, $c = m + n$

(c) $a = xy$, $b = xz$, $c = (?)$

3. Given $\sin A = \frac{5}{7}$, construct the angle A, and measure it with a protractor.

4. Given $\cos A = \frac{2}{3}$, construct and measure the angle.

5. Given $\tan A = \frac{1}{3}$, construct and measure the angle.

6. Given $\cos A = \frac{4}{5}$, find the other functions of A.

7. Given $\sec A = \frac{13}{5}$, find the other functions.

8. Given $\operatorname{ctn} A = \frac{8}{7}$, find the other secondary functions of A.

9. Construct carefully with a protractor angles of 15°, 20°, 25°, 30°, 60°. Measure the sides of a triangle of reference for these angles, and make a table of sines, cosines, and tangents of each angle. Compare these values with those given in the table in § 4, and tabulate the exact differences between your results and those of the table.

10. Using the table thus formed, find the value of a and b in a right triangle ABC if $A = 20°$, $C = 90°$, $c = 12$.

11. Using the table of Exercise 9, find the values of c and b in a triangle ABC in which $B = 25°$, $a = 42$, $C = 90°$.

12. Find, from the table of Exercise 9, the values of c and a in a right triangle ABC in which $B = 70°$, $b = 20$, $C = 90°$.

2. Functions of the Complementary Angle.

1. Express as functions of the complementary angle:

(a) sin 30° (c) tan 40° (e) sec 38°

(b) cos 20° (d) ctn 25° (f) csc 17°

(*g*) tan 3° (*k*) sin 12.15° (*o*) sin 90°
(*h*) cos 5° 12′ (*l*) sec 22.8° (*p*) sec 45°
(*i*) cos 50.17° (*m*) cos 41° 10′ 8″ (*q*) csc 54.02°
(*j*) ctn 87° 10′ (*n*) tan 13° 4′ 3″ (*r*) ctn 1.037°

2. Express as functions of an angle less than 45°:

(*a*) cos 73° (*d*) ctn 87.35° (*g*) sin 67° 21′
(*b*) tan 87° (*e*) sec 85.02° (*h*) cos 55° 2′ 3″
(*c*) sin 60° (*f*) csc 89.03° (*i*) tan 70° 5′ 21″

3. Find a value of x for which

(*a*) $\sin x = \cos x$ (*e*) $\sin x = \cos 3x$
(*b*) $\tan x = \operatorname{ctn} x$ (*f*) $\sin 2x = \cos (45° - x)$
(*c*) $\tan x = \operatorname{ctn} (30° + x)$ (*g*) $\sec x = \csc 4x$
(*d*) $\operatorname{ctn} \frac{1}{2} x = \tan x$ (*h*) $\cos x = \sin (n - 1)x$

3. Functions of 30°, 45°, and 60°.

1. Find the numerical values of

(*a*) $2 \sin 30° \cos 30° \operatorname{ctn} 60°$
(*b*) $\tan^2 45° + 4 \cos^2 60°$
(*c*) $\sin 60° \cos 30° + \cos^2 60°$
(*d*) $\sin^2 45° - \tan 45° + \cos^2 45°$

2. Show that $\tan 60° = \dfrac{2 \tan 30°}{1 - \tan^2 30°}$.

3. Show that $\cos 60° = 2 \cos^2 30° - 1$.

4. Show that $\sin^2 30° = \dfrac{1 - \cos 60°}{2}$.

5. Show that $\sin^2 45° = \cos 60°$.

6. Show that $\sec^2 45° + \tan^2 45° = \operatorname{ctn}^2 30°$.

7. Show that $(\csc 45° + \sec 45°)^2 = \csc^2 30° \sec 60°$.

8. Find the value of $\sin 30° \cos 60° + \cos 30° \sin 60°$.

9. Show that $\sin 60° \cos 30° - \cos 60° \sin 30° = \sin 30°$.

10. Show that $\tan 30° = \dfrac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°}$.

11. Show that $\cos 30° = \cos 60° \cos 30° + \sin 60° \sin 30°$.

4. Trigonometric Tables.

1. Find by means of the table of § 4:

(*a*) sin 18° (*c*) tan 37° (*e*) cos 72° (*g*) sec 12°
(*b*) sin 82° (*d*) tan 16° (*f*) cos 41° (*h*) csc 77°

2. Find the angles whose functions have the following values, using the table of § 4:

(*a*) $\sin x = 0.208$ (*c*) $\cos x = 0.616$ (*e*) $\tan x = 0.404$
(*b*) $\sin x = 0.978$ (*d*) $\cos x = 0.993$ (*f*) $\tan x = 0.810$

3. Find the angle A, by means of the table of § 4, from the following data:

(*a*) $\sec A = 1.07$ (*c*) $\csc A = 1.11$ (*e*) $\sin A = 0.988$
(*b*) $\sec A = 1.02$ (*d*) $\csc A = 1.05$ (*f*) $\cos A = 0.988$

4. Round off the following quantities into numbers correct to three significant figures:

(*a*) 3.14159 (*d*) 2.71828 (*g*) 0.7036 (*j*) 47850
(*b*) 1.414 (*e*) 2.2046 (*h*) 9297.9 (*k*) 243600
(*c*) 39.35 (*f*) 1609.35 (*i*) 3.265 (*l*) 0.008256

In multiplying or dividing numbers which are correct to n significant figures, only the first n significant figures should be retained in the results.

5. Find the product of 3.142 by 1.414 correct to four significant figures.

6. Find the product, correct to four significant figures, of 2.718 by 2.205 by 0.07036.

7. Find the value, correct to three significant figures, of $1609 \div 62.40$.

8. Compute the value, correct to three significant figures, of

$$\frac{1375 \times 0.06423}{176.42}.$$

9. Compute the value, correct to three significant figures, of

$$\frac{928 \times 0.935}{1278}.$$

10. Compute the cube of 17.28 correct to four significant figures.

5. Four-place Tables.

1. Find the tangent and cotangent of the following angles:

(*a*) 34.6° (*c*) 87.3° (*e*) 24.32° (*g*) 42.28°
(*b*) 63.9° (*d*) 53.4° (*f*) 56.73° (*h*) 73.67°

2. Find the sine and cosine of the following angles:

(*a*) 10.32° (*c*) 75.43° (*e*) 5.34°
(*b*) 57.37° (*d*) 84.92° (*f*) 0.73°

3. Find the secant and cosecant of the following angles:

(*a*) 32.84° (*c*) 79.62° (*e*) 1.86°
(*b*) 60.27° (*d*) 85.43° (*f*) 14.75°

4. Find the values of the following functions:

(*a*) tan 35.93°	(*c*) sec 64.28°	(*e*) ctn 4.57°
(*b*) cos 24.37°	(*d*) sin 89.82°	(*f*) csc 87.12°

5. Find x when

(*a*) $\sin x = 0.4939$	(*d*) $\sin x = 0.4960$	(*g*) $\tan x = 0.3983$
(*b*) $\cos x = 0.3387$	(*e*) $\sec x = 1.573$	(*h*) $\csc x = 1.6396$
(*c*) $\tan x = 2.184$	(*f*) $\cos x = 0.3773$	(*i*) $\operatorname{ctn} x = 0.4797$

6. Solution of Right Triangles.

1. Solve the following right triangles ($C = 90°$) graphically and check your results by use of the tables:

(*a*) $A = 30°, b = 8$	(*d*) $B = 40°, b = 5$	(*g*) $a = 7, b = 4$
(*b*) $A = 45°, c = 14$	(*e*) $A = 75°, a = 12.5$	(*h*) $b = 15, c = 27$
(*c*) $b = 9, c = 20$	(*f*) $B = 24°, c = 13$	(*i*) $a = 4, b = 12$

Solve the following right triangles by means of the tables of trigonometric functions ($C = 90°$):

	Given		Required		
2.	$A = 36.00°$	$c = 2.000$	$B = 54.00°$	$a = 1.176$	$b = 1.618$
3.	$B = 57.00°$	$c = 10.00$	$A = 33.00°$	$a = 5.446$	$b = 8.387$
4.	$A = 42.40°$	$c = 12.00$	$B = 47.60°$	$a = 8.092$	$b = 8.862$
5.	$a = 5.000$	$c = 10.00$	$A = 30.00°$	$B = 60.00°$	$b = 8.660$
6.	$a = 3.000$	$b = 3.000$	$A =$	$B =$	$c =$
7.	$a = 28.23$	$c = 100.0$	$A = 16.40°$	$B = 73.60°$	$b = 95.93$
8.	$b = 9.265$	$c = 10.00$	$A =$	$B =$	$a =$
9.	$a = 3.000$	$b = 4.000$	$A = 36.87°$	$B = 53.13°$	$c = 5.000$
10.	$A = 75.00°$	$a = 80.00$	$B = 15.00°$	$b = 21.43$	$c = 82.82$
11.	$a = 5.000$	$c = 13.00$	$A =$	$B =$	$b =$
12.	$B = 86.00°$	$b = 0.0800$	$A = 4.00°$	$a = 0.00559$	$c = 0.0802$
13.	$a = 21.15$	$b = 50.00$	$A = 22.93°$	$B = 67.07°$	$c = 54.29$
14.	$a = 7.977$	$c = 30.00$	$A =$	$B =$	$b =$
15.	$B = 32.00°$	$c = 1760$	$A = 58.00°$	$b = 932.6$	$a = 1492$
16.	$A = 35.00°$	$b = 196.6$	$B =$	$a =$	$c =$
17.	$a = 6.000$	$c = 103.0$	$A =$	$B =$	$b =$
18.	$c = 59.00$	$A = 38° 29'$	$B = 51° 31'$	$a = 36.71$	$b = 46.19$
19.	$A = 84° 52'$	$b = 0.2800$	$B =$	$a =$	$c =$
20.	$B = 46° 11'$	$a = 191.9$	$A =$	$b =$	$c =$

Find the areas of the following right triangles:

21. $A = 34.22°, b = 14.31.$ *Ans.* 69.63

22. $B = 15.28°$, $c = 14.98$.

23. $a = 17.84$, $c = 41.26$. *Ans.* 331.8.

24. $B = 59.27°$, $b = 4.182$.

25. Show that the area of a right triangle expressed in terms of A and c equals $\frac{1}{4} c^2 \sin A \cos A$.

26. Express the area of a right triangle in terms of A and b.

27. From the point A on one side of a body of water a point B is sighted on the other side. The line AC is laid off 500 feet long at right angles to the line of sight AB. The angle ACB measures 76.64°. Find the distance AB across the water. *Ans.* 2106 feet.

28. A tree casts a shadow 67.5 feet long on a horizontal plain, when the sun is 54.32° above the horizon. Find the height of the tree.

29. A building is 56.30 feet high, and fronts on a street 37.25 feet wide. If the sun is directly behind the building, what angle must it reach above the horizon before the shadow of the building will fail to cover the width of the street?

The *angle of elevation* or the *angle of depression* of an object is the angle which a line from the eye to the object makes with a horizontal line in the same vertical plane.

30. Find the angle of elevation of a tower which is 127 feet high and is 643 feet away on a horizontal field. *Ans.* 11.17°.

31. What is the height of a mountain which is 5.50 miles away from the observer, if the angle of elevation of its summit is 15.27°?

32. A man is facing backward in a train with his head close to one side of the car. In front of him is a mirror, in the end of the car, 3.5 feet away, and directly across from him is a window. He sees the image of a mountain peak in the mirror, and 250 seconds later sees the peak itself by looking straight across the car out the window. The train is traveling 30 miles an hour, and the image appears in the mirror 3 feet from his side of the car. Find the distance of the peak from the track. *Ans.* about 9400 feet.

The bearing of a point P from a given point A is the direction of the line AP. Thus if P is northeast of A, we say P bears N. E. from A. If P is 20° east of north from A, we say P bears N. 20° E. from A.

33. A certain lighthouse is due west from a ship. A second light which is known to be 12.8 miles due north of the first bears N. 25.7° W. from the ship. How far is the ship from the first light?

34. Two mountain peaks are, one due north and the other due west from a balloon. If the first peak is 39.5 miles from the balloon, and the second peak bears S. 32.4° W. from the first, what is the distance between the peaks?

35. From a second-story window, the top of a building across the street has an angle of elevation of 47.20°, while its base has an angle of depression of 29.30°. If the street is 43.4 ft. wide, what is the height of the building? *Ans.* 71.2 feet.

36. A man stands on a cliff so that his eye is 42 feet above the water. He observes the angle of depression of the water line of a boat to be 1.25°. How far away is the boat?

37. An aeroplane, M, is observed from two stations, A and B, which are 6728 feet apart, and P is a point directly below M on the same horizontal plane with A and B. At A the horizontal angle between AB and AP is 35°, and the angle of elevation of M is 27°. At B the horizontal angle between BA and BP is 55°, and the angle of elevation of M is 36.° Find the height of the aeroplane as observed from each station. *Ans.* 2804 feet; 2808 feet.

38. A man at point A on a beach observes that the angle of elevation of a mountain peak on an island is 40°, and that the peak bears directly east. At a second point, B, on the beach, 4200 feet from A, he observes that the angle of elevation is 27°, and that the peak bears due north. Find the altitude of the peak above sea level, and the distance from A to the peak.

39. From a point A on a level plain, the angle of elevation of a mountain peak 6000 feet above the plain is 75°. A cloud from which rain is falling upon the peak has an angle of elevation of 78° from the same point. How high is the cloud above the peak?

40. From a ship sailing directly toward an island the angle of elevation of the highest peak of the island is observed to be 6°. After sailing one mile on the same course, the angle is again taken and found to be 10°. Find the height of the peak.

41. A railroad track on a plain runs due north. From a certain position, an observer sees two stations which he knows to be two miles apart. The first bears N. 35° E. and the second bears N. 25° E. from the observer. Find the shortest distance from the observer to the track.

42. A certain road between A and B rises 1 foot for every horizontal advance of 8 feet. What will be the rise in going from A to B if the actual distance traveled along the road is 9.37 miles?

43. A certain inclined plane rises 3.25 feet in a horizontal distance of 35.5 feet. Find the angle which it makes with the horizontal, and the length of the incline for a rise of 27.2 feet.

An isosceles triangle may be divided into two equal right triangles by drawing a perpendicular from the vertex to the base. When any two of the four parts of the isosceles triangle are given, all the other parts can be computed by means of these right triangles. A regular polygon may be treated by dividing it into isosceles triangles.

44. Find the remaining parts of an isosceles triangle if one of the equal sides is 273.5, and the vertical angle is 64.00.°

45. Each of the base angles of an isosceles triangle is 27.13°, and each of the equal sides is 3.088. Find the remaining parts.

46. Two lights stand on opposite points of land 4750 feet apart at the entrance of a harbor. One light is directly north of the other, and a ship is due east of a point half way between them. The north light bears W. 27.35° N., and the other, W. 27.35° S. from the ship. How far must the ship sail due west to come directly between the lights?

47. Find the length of the apothem of a regular octagon inscribed in a circle whose radius is 25.00.

48. Find the difference between the perimeters of a regular inscribed hexagon and a regular inscribed octagon if the radius of the circle is 10.

7. Projections.

1. Two straight lines, AB and CD, make an angle x with each other. Find the projection on CD of a segment, m, of AB when

(*a*) $x = 37.28°$, $m = 14.92$ in. (*b*) $x = 82.43°$, $m = 0.317$ in. (*c*) $x = 46.26°$, $m = 187.3$ in.

2. If the projection of the segment m of AB on CD is p, what angle do the two lines make when

(*a*) $m = 144.3$, $p = 27.3$ (*b*) $m = 52.57$, $p = 13.62$ (*c*) $m = 185.9$, $p = 132.5$

3. An air-ship 253 feet long casts a shadow 245 feet long when the sun is directly overhead. At what angle is it inclined to the horizon?

Ans. 14.44°

4. The back of a Morris chair inclines 22.5° from the vertical, and is 28 inches long. How far from the wall must the chair legs stand, if they are 5 inches nearer the wall than the lower end of the back?

8. Line Values of the Functions.

1. If x is an acute angle, show by means of the line values, and also by means of the definitions of the functions, that

(*a*) $\tan x > \sin x$ (*c*) $\csc x > \operatorname{ctn} x$

(*b*) $\cos x < \operatorname{ctn} x$ (*d*) $\sin x < 2 \sin \frac{x}{2}$

2. Construct the angle x by means of a unit circle when

(*a*) $\tan x = 2$ (*c*) $\cos x = 0.75$ (*e*) $\sin x = \cos x$

(*b*) $\sin x = 0.5$ (*d*) $2 \sin x = \tan x$ (*f*) $3 \cos x = \operatorname{ctn} x$

3. Construct the line value of $\sin 60° + \cos 30°$ by means of a unit circle.

4. Construct the line value of $3 \cos 30°$, and compare it with $\cos 90°$.

5. Construct $\sin (x + y)$ and $\sin x + \sin y$ by means of a unit circle, x, y, and $(x + y)$ all being definite acute angles.

9. Trigonometric Identities.

Prove that

1. $\cos A \tan A = \sin A$.
2. $\sin A \operatorname{ctn} A = \cos A$.
3. $\sin A \sec A \operatorname{ctn} A = 1$.
4. $\tan A \cos A - \sin A = 0$.
5. $\dfrac{\sin A}{\csc A} + \dfrac{\cos A}{\sec A} = 1$.
6. $(\tan^2 x + 1) \cos^2 x = \sin^2 x + \cos^2 x$.
7. $\tan^2 x \cos^2 x = \sin^2 x \csc^2 x - \cos^2 x$.
8. $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2 \csc x$.
9. $\operatorname{ctn} x + \dfrac{\sin x}{1 + \cos x} = \csc x$.
10. $\dfrac{1 + \cos B}{1 - \cos B} = (\csc B + \operatorname{ctn} B)^2$.
11. $\dfrac{1 - \operatorname{ctn}^2 B}{1 + \operatorname{ctn}^2 B} = \sin^2 B - \cos^2 B$.

10. Trigonometric Equations.

Solve the following equations:

1. $\sin^2 x = \frac{3}{4}$. *Ans.* $x = 60°$.
2. $\operatorname{ctn}^2 x = \frac{1}{3}$. *Ans.* $x = 60°$.

3. $\tan x + \operatorname{ctn} x = 2$. *Ans.* $x = 45°$.

4. $2 \operatorname{ctn}^2 y + \operatorname{ctn} y = 6$.

5. $2 \cos y + \sec y = 3$.

6. $\cos^2 y = \sin^2 y + \sin y$.

7. $2 \tan y - \sec y = 0$.

8. $2 \sin y = 4 \cos y$.

9. $\sqrt{3} \sin A = \cos A$.

10. $\sin A - \sqrt{3} \cos A = 0$.

11. $\sin^2 A + \sin A = \frac{3}{4}$. *Ans.* $A = 27.22$.

12. $\cos A + \sec A = 2$.

13. $6 \sin A - 3 \csc A = 2$.

14. $\tan^2 x - 3 \tan x = 5$.

15. $10 \cos^2 x + 3 = 11 \cos x$.

CHAPTER II

11. Logarithms. Definitions and Fundamental Laws.

1. Give the value of each of the following: log 100, log 10,000, log 0.01, log 100,000, log $\frac{1}{100}$, log $\frac{1}{10}$, log 0.001, log 100,000,000, log 0.00001, log $\frac{1}{10000}$.

2. Find the numbers of which each of the following is the logarithm: 2, 7, -1, 5, -3, 8, -2, 4, -4, 3.

3. Find the values of

(*a*) $\log 100^8$ (*e*) $\log 1000^3$ (*i*) $\log (\frac{1}{1000})^3$
(*b*) $\log 10^5$ (*f*) $\log (10{,}000)^2$ (*j*) $\log (\frac{1}{10})^6$
(*c*) $\log (0.01)^7$ (*g*) $\log (0.0001)^3$ (*k*) $\log 10^{15}$
(*d*) $\log (0.001)^2$ (*h*) $\log (\frac{1}{100})^7$ (*l*) $\log (0.1)^8$

4. Find the values of

(*a*) $\log \sqrt{1000}$ (*d*) $\log \sqrt[5]{100}$ (*g*) $\log \sqrt[7]{10{,}000}$
(*b*) $\log \sqrt[3]{100}$ (*e*) $\log \sqrt[5]{10}$ (*h*) $\log \sqrt[3]{0.0001}$
(*c*) $\log \sqrt{0.001}$ (*f*) $\log \sqrt[4]{1000}$ (*i*) $\log \sqrt{0.1}$

5. Show that

(*a*) $\log \sqrt{1000} + \log 100 = 3\frac{1}{2}$
(*b*) $\log 1000 - \log (\frac{1}{10}) + \log \sqrt[5]{10{,}000} = 4.8$
(*c*) $\log \sqrt[5]{1000} + \log 0.001 - \log \sqrt{100} = -3.4$

(*d*) $\log \sqrt{\frac{1}{100}} + \log \sqrt[3]{100} - \log 100 = -\frac{7}{6}$

(*e*) $\log (0.1)^3 - \log \sqrt[4]{100} = -\frac{7}{2}$

6. Write the following expressions as the algebraic sums of logarithms:

(*a*) $\log \frac{a \times b \times c}{d \times e}$

(*b*) $\log \left(\frac{bc \sin A}{2}\right)$

(*c*) $\log \left[\frac{(s-a)(s-b)(s-c)}{s}\right]$

(*d*) $\log \left(\frac{\sqrt{m}\sqrt[3]{n}}{pr}\right)$

7. Write in expanded form

(*a*) $\log \sqrt[4]{\frac{x^2(y-z)}{2\sqrt[3]{d}}}$

(*b*) $\log \sqrt[5]{\frac{m\sqrt{x-y}}{4ab}}$

(*c*) $\log \frac{(x+y)(a-b)}{\sqrt[3]{a(a+b)}}$

(*d*) $\log \sqrt[n]{2ab \sin c}$

8. Write in contracted form

(*a*) $3 \log a + \frac{1}{2} \log b - \log (a-b)$.

(*b*) $5 \log (m-n) + 4 \log c - \log m - 2 \log n$.

(*c*) $2 \log (1-x) + \frac{1}{2} \log y - \log x - \frac{3}{4} \log (2-x)$.

(*d*) $6 \log 5 + \frac{3}{4} \log 8 + \frac{1}{4} \log 6 - \log 9 - \log 2 - \frac{1}{2} \log 7$.

(*e*) $\frac{1}{2} (\log 8 + 4 \log 245 + \log 165 - \frac{3}{4} \log 186 - \log 259)$.

12. Logarithmic Tables.

1. Find the logarithms of the following numbers:

(*a*) 2.386	(*e*) 1258	(*i*) 85962
(*b*) 49.36	(*f*) 0.07629	(*j*) 73.8763
(*c*) 0.8325	(*g*) 0.0005472	(*k*) 3.14159
(*d*) 985.4	(*h*) 4327000	(*l*) 1.732537

2. Find the numbers whose logarithms are:

(*a*) 1.4713	(*e*) $\bar{3}$.9761	(*i*) 2.7241
(*b*) 3.8845	(*f*) $\bar{2}$.2434	(*j*) 0.3011
(*c*) 0.5919	(*g*) 4.9118	(*k*) 6.2931
(*d*) $\bar{1}$.7348	(*h*) 3.9236	(*l*) 9.8738

3. Find the cologarithms of:

(*a*) 273.0	(*d*) 0.01256	(*g*) 1.796	(*j*) 0.001507
(*b*) 45.73	(*e*) 9963	(*h*) 837.2	(*k*) 0.4398
(*c*) 0.9134	(*f*) 5739	(*i*) 0.005283	(*l*) 643.2

13. Computation by Means of Logarithms.

Compute by means of logarithms:

1. 321.5×9.738.
2. 7931×0.05864.
3. 3645×2716.
4. 2001×58.23.
5. 8795×0.003684.
6. 5.013×0.0005827.
7. 2395×0.5003.
8. 8032×971.6.
9. $2035 \div 1938$.
10. $3721 \div 23.23$.
11. $6.666 \div 1283$.
12. $7.391 \div 23.33$.
13. $5.111 \div 888.8$.
14. $1.915 \div 182.7$.
15. $9.126 \div 635.2$.
16. $100.2 \div 19999$.
17. $0.0001351 \div 0.03827$.
18. $0.003999 \div 9993$.
19. 7.221×722.1.
20. 1.838×0.0008888.
21. 0.001212×2121.
22. 0.005931×9999.
23. 0.1639×0.1073.
24. 9732×0.009732.
25. $2567 \times 8467 \times 9472 \times 7309$.
26. $9006 \times 3860 \times 0.7346 \times 0.003562$.
27. $0.0008947 \times 8999 \times 856.2$.
28. $2929 \div 4466 \times 9784 \div 9572$.
29. $25.73 \div (5.800 \times 0.6843 \times 86.24)$.
30. $(7385)^4$.
31. $(0.07385)^7$.
32. $(82.55)^{\frac{3}{4}}$.
33. $\sqrt[3]{8452}$.
34. $\sqrt[5]{2.006}$.
35. $\dfrac{731 \times 4.386}{1.742 \times 14.82}$.
36. $\dfrac{4807 \times 9.532 \times 4.444}{1.795 \times 58.39 \times 70.33}$.
37. $\dfrac{0.005693 \times 835.7}{6.573 \times 79.26}$
38. $\dfrac{99.99 \times 45.45 \times 2.200}{68.62 \times 9555 \times 1101}$.
39. $\dfrac{88.88 \times 3636 \times 9285}{4684 \times 212.0 \times 8.333}$.
40. $\sqrt[4]{1.729} \times \sqrt[3]{2391}$.
41. $\sqrt[7]{68.45} \times \sqrt{7387}$.
42. $\dfrac{\sqrt[3]{389.2} \times \sqrt[4]{0.1737}}{(8540)^2 \times 0.9136}$.
43. $\sqrt{\dfrac{231.7 \times 0.0001542}{(3.427)^2 \times 2843}}$.
44. $\dfrac{(3058)^{\frac{1}{2}} \times (-573.2)}{(-5429) \times (-0.02315)}$
45. $\dfrac{3.725\,\pi}{\sqrt[7]{897300}}$
46. $\dfrac{(6391)^3 \times 3643\,\pi}{\sqrt[4]{\pi}\,(3487)}$.
47. $\left(\dfrac{2.956}{3582}\right)^{-4}$
48. $\left(\dfrac{5.103}{663.6 \times 8.791}\right)^{-\sqrt{2}}$.

49. Find x from the following equations:

(*a*) $x = 0.1732^{0.3251}$ (*b*) $x = 0.05273^{0.7138}$

50. Solve for x:

(*a*) $x + 2 = 0.01302^{0.03582}$ (*b*) $7x = 0.003795^{0.05432}$

51. Solve for x:

(*a*) $4^x = 9$ (*e*) $8^{x+2} = 4$

(*b*) $10^x = 3$ (*f*) $(2.7)^{x-4} = 1.8$

(*c*) $85^x = 0.35$ (*g*) $3^{3x+2} = 20$

(*d*) $55^x = 1$ (*h*) $26^{5x-1} = 405$

14. Logarithms with Other Bases.

1. Find the values of the following logarithms:

(*a*) $\log_2 8$ (*g*) $\log_2 (\frac{1}{16})$ (*m*) $\log_{0.5} 8$

(*b*) $\log_5 125$ (*h*) $\log_2 9$ (*n*) $\log_{0.3} 0.1$

(*c*) $\log_3 27$ (*i*) $\log_3 16$ (*o*) $\log_{0.8} 1793$

(*d*) $\log_4 64$ (*j*) $\log_7 12$ (*p*) $\log_{2.718} 1543$

(*e*) $\log_3 (\frac{1}{9})$ (*k*) $\log_{2.5} 18$ (*q*) $\log_{2.718} 57.29$

(*f*) $\log_5 (\sqrt[3]{\frac{1}{25}})$ (*l*) $\log_5 0.01$ (*r*) $\log_{3.678} 7.521$

2. Show that $\log_2 32 + \log_7 49 + \log_{10} 10000 - \log_4 64 = 8$.

3. Find the base of the system in which $\log 9 = \frac{2}{3}$.

4. Find the base if $\log 73 = \frac{3}{5}$.

5. Construct a table of the powers of 3 from the -4th to the $+6$th.

6. Find by means of the table computed in Exercise 5:

(*a*) 81×9 (*b*) $\frac{1}{81} \times 27$ (*c*) $\frac{1}{3} \times 81$

7. Perform the following divisions by means of the table of Exercise 5:

(*a*) $243 \div 27$ (*b*) $729 \div 9$ (*c*) $81 \div (\frac{1}{9})$ (*d*) $9 \div (\frac{1}{81})$

8. Find x in the following formulas:

(*a*) $\log_3 x = 4$ (*c*) $\log_4 x = 16$ (*e*) $\log_{10} x = 10$

(*b*) $\log_2 x = 6$ (*d*) $\log_8 x = 51$ (*f*) $\log_9 x = 65$

9. Prove that $(\log_b N)(\log_N b) = 1$.

10. Given $\log_{10} 5 = 0.6990$, find $\log_5 10$.

15. Solution of Right Triangles by Means of Logarithms.

1. Find the values of the following expressions:

(*a*) log sin 21.60° (*f*) log tan 52.75°

(*b*) log sin 42.78° (*g*) log tan 64.83°

(*c*) log sin 3.214° (*h*) log tan 7.44°

(*d*) log sin 82.78° (*i*) log tan 74.28°

(*e*) log sin 88.61° (*j*) log tan 84.71°

2. Find the angle x when

(*a*) $\log \sin x = \bar{1}.6197$
(*b*) $\log \sin x = \bar{1}.8223$
(*c*) $\log \sin x = \bar{1}.9556$
(*d*) $\log \sin x = \bar{1}.9978$
(*e*) $\log \sin x = \bar{1}.1996$
(*f*) $\log \sin x = \bar{1}.1647$
(*g*) $\log \sin x = \bar{2}.8853$
(*h*) $\log \sin x = \bar{1}.2197$
(*i*) $\log \sin x = \bar{1}.0005$
(*j*) $\log \sin x = \bar{1}.9896$
(*k*) $\log \tan x = 2.5439$
(*l*) $\log \tan x = \bar{2}.8529$
(*m*) $\log \tan x = \bar{2}.4855$
(*n*) $\log \tan x = \bar{1}.2003$
(*o*) $\log \tan x = 0.2497$
(*p*) $\log \tan x = 0.9759$
(*q*) $\log \tan x = 1.0247$
(*r*) $\log \tan x = 1.7027$
(*s*) $\log \tan x = 0.3049$
(*t*) $\log \tan x = \bar{1}.9640$

3. Find the values of the following expressions:

(*a*) $\log \cos 23.13°$
(*b*) $\log \cos 63.98°$
(*c*) $\log \cos 83.61°$
(*d*) $\log \operatorname{ctn} 41.72°$
(*e*) $\log \operatorname{ctn} 71.38°$
(*f*) $\log \operatorname{ctn} 86.72°$
(*g*) $\log \operatorname{ctn} 6.47°$
(*h*) $\log \cos 84.97°$
(*i*) $\log \operatorname{ctn} 88.24°$
(*j*) $\log \operatorname{ctn} 80.79°$

4. Find the values of x for which

(*a*) $\log \cos x = \bar{2}.4728$
(*b*) $\log \operatorname{ctn} x = 0.2745$
(*c*) $\log \operatorname{ctn} x = 0.9787$
(*d*) $\log \cos x = \bar{1}.9905$
(*e*) $\log \cos x = \bar{2}.4906$
(*f*) $\log \cos x = \bar{1}.1903$
(*g*) $\log \operatorname{ctn} x = \bar{2}.8775$
(*h*) $\log \operatorname{ctn} x = 1.0247$
(*i*) $\log \operatorname{ctn} x = 0.9999$
(*j*) $\log \cos x = \bar{1}.9999$

5. Find the values of the following expressions:

(*a*) $\log \sin 0.125°$
(*b*) $\log \sin 0.072°$
(*c*) $\log \cos 89.863°$
(*d*) $\log \tan 0.244°$
(*e*) $\log \tan 0.021°$
(*f*) $\log \operatorname{ctn} 89.615°$
(*g*) $\log \operatorname{ctn} 0.732°$
(*h*) $\log \tan 89.853°$
(*i*) $\log \tan 0.023°$
(*j*) $\log \cos 88.967°$

6. Find x when

(*a*) $\log \sin x = \bar{4}.5433$
(*b*) $\log \cos x = \bar{2}.1459$
(*c*) $\log \tan x = \bar{3}.4465$
(*d*) $\log \operatorname{ctn} x = 2.1567$

Solve the following right triangles ($C = 90°$):

No.	Given		Required		
7.	$a = 153.0$	$c = 306.0$	$A = 30.00°$	$B = 60.00°$	$b = 265.1$
8.	$a = 51.72$	$b = 215.8$	$A = 13.48°$	$B = 76.52°$	$c = 221.8$
9.	$A = 32.17°$	$c = 4.185$	$B = 57.83°$	$a = 2.228$	$b = 3.542$
10.	$B = 53.42°$	$c = 6500$	$A = 36.58°$	$a = 3874$	$b = 5219$
11.	$a = 372.9$	$B = 15.46°$	$A =$	$b =$	$c =$

No.	Given		Required		
12. $a = 9.518$	$A = 35.41°$	$B = 54.59°$	$b = 13.38$	$c = 16.43$	
13. $b = 5973$	$A = 82.16°$	$B =$	$a =$	$c =$	
14. $b = 3.780$	$B = 12.97°$	$A = 77.03°$	$a = 16.41$	$c = 16.84$	
15. $A = 86° 40'$	$a = 4.682$	$B = 3° 20'$	$b = 0.2726$	$c = 4.689$	
16. $B = 6.35°$	$a = 27190$	$A =$	$b =$	$c =$	
17. $A = 0.253°$	$b = 3512$	$B = 89.75°$	$a = 15.51$	$c = 3512$	
18. $A = 87° 53'$	$a = 27.51$	$B =$	$b =$	$c =$	
19. $A = 89.46°$	$c = 0.03719$	$B =$	$a =$	$b =$	
20. $B = 47° 12'$	$a = 1.793$	$A =$	$b =$	$c =$	
21. $c = 0.5901$	$a = 0.06382$	$A =$	$B =$	$b =$	
22. $B = 0.163°$	$c = 51.78$	$A =$	$a =$	$b =$	
23. $a = 0.004527$	$b = 0.05638$	$A = 4.59°$	$B = 85.41°$	$c = 0.05658$	
24. $c = 349300$	$a = 20930$	$A =$	$B =$	$b =$	
25. $b = 5.93824$	$c = 819.768$	$A =$	$B =$	$a =$	
26. $A = 59° 35'$	$b = 40.008$	$B =$	$a =$	$c =$	

27. The top of a lighthouse is known to be 127 feet above the water, and it has an angle of elevation of 6° 21′ as seen from a small rowboat. How far away is the boat?

28. When the angle of elevation of the sun is 67.26°, the shadow of a certain tree on a level field is 47.20 feet long. How high is the tree?

Ans. 112.6 feet.

29. How much string, allowing for no sag, will be required to fly a kite from one edge of a river 327 feet wide so that the kite itself shall be directly over the opposite bank when its angle of elevation is 63°?

30. A ladder 40 feet long stands against a building. How high can it reach if the greatest safe angle which it can make with the ground is 80°?

Ans. 39.4 feet.

31. What angle must a ladder 45 feet long make with the ground to reach a window-sill 43.75 feet from the ground?

32. An observer is stationed at a distance of 7380 feet from a fort. What will be the least angle of elevation that an aeroplane directly over the fort will have at the observer's station when it has descended to a position not lower than 2000 feet above the fort?

Ans. 15.16°.

33. The top of a mountain is 5225 feet above the base. What is the least grade which a road to the top can have, if it is to be not longer than 10.2 miles?

34. From a point A the angle of elevation of the top of a hill is 49.23°. From a point B, 1000 feet farther away in the same direction on

a level plain, the angle of elevation of the top is 26.17°. How high is the top of the hill above the plain?

35. From station A, the angles of depression of the top of a mast and the water line of a ship are, respectively, 19.26° and 23.87°. By moving to station B, 1000 feet farther away from the ship, the two angles become, respectively, 9.92° and 12.49°. Find the height of the mast top above the water. *Ans.* 93.3 feet.

36. The town A is 20.18 miles directly north of B, and C is 36.05 miles directly west of B. What is the direction of the road from C to A?

37. The town A is 22 miles N. E. of B, and is separated from it by a lake. There are two possible routes for a road from A to B. First, from A go north 70° W. to meet a road which runs N. 20° E. from B. Second, go south 16° W. to meet a road which runs S. 74° E. from B. What is the length of the shorter of the two routes?

Ans. 29.24 miles.

38. A lighthouse L is directly north of a second light K. From a certain position of a ship, L bears S. 47° E, and K bears S. 38° E. After the ship has sailed due south 55 miles, L bears N. 43° E. Find the distance between the lights, and the shortest distances from each light to the ship.

39. A pulley 12 inches in diameter and another 20 inches in diameter are running on shafts 72 inches apart. Find the length of belt needed to connect up the two pulleys, if the belt is not crossed.

Ans. 194 inches.

40. Two pulleys 8 inches and 24 inches in diameter are belted with a crossed belt, on shafts which are 80 inches apart. Find the length of belt required.

41. Find the altitude of an isosceles triangle when the legs are each 32.57 feet and the base angles are each 78.32°.

42. Find the legs of an isosceles triangle when the base is 20.16 feet, and the base angles are each 73.28°. *Ans.* 35.04 feet.

43. Find the base angles of an isosceles triangle in which the base is 72.50 feet and the legs are each 43.18 feet. *Ans.* 32.92°.

44. Find the vertical angle of an isosceles triangle in which the base is 34.29 feet and the altitude 15.43 feet.

45. Two lights with a range of 36.75 miles each are to be established on a coast. How far apart can they be placed so that a ship sailing parallel to the coast, and 24.25 miles off shore, will not lose sight of one before the other comes into view? *Ans.* 55.24 miles.

46. A ship has wireless apparatus with a range of 200 miles. If wireless stations are placed along the coast at intervals of 300 miles, how far off shore can a ship sail parallel to the coast and not be out of communication with one or another of the stations?

47. Find the length of the side of a regular pentagon inscribed in a circle of which the radius is 8.157.

48. Find the apothem of a regular octagon inscribed in a circle of radius 12.20 inches. *Ans.* 11.27 inches.

49. A regular polygon of 12 sides and area 7624 square inches is inscribed in a circle. Find the radius of the circle.

50. Find the difference between the areas of a regular hexagon and a regular pentagon inscribed in a circle of which the radius is 7 inches.

Ans. 10.80 square inches.

51. What is the radius of a circle in which a chord 100 feet long subtends an arc of 43°?

52. A certain wheel 4.28 feet in diameter is to have 27 spokes. At what distance apart (in a straight line) should the holes be bored in the rim of the wheel to receive the ends of the spokes? *Ans.* 0.50 feet.

CHAPTER III

16. Angles of Any Magnitude.

1. In what quadrant do the following angles lie?

(*a*) 140°	(*f*) 375°	(*k*) 1000°	(*p*) 359°
(*b*) 320°	(*g*) − 490°	(*l*) − 700°	(*q*) 721°
(*c*) 65°	(*h*) 920°	(*m*) 269°	(*r*) − 179°
(*d*) − 80°	(*i*) 750°	(*n*) − 91°	(*s*) 271°
(*e*) 210°	(*j*) 800°	(*o*) 0.5°	(*t*) 899°

2. For what values of x will the angle $(90° - x)$ lie in the first quadrant?

3. For what values of x will the angle $(180° + x)$ lie in the second quadrant?

4. For what values of x may the angle $(90° + x)$ be one of the angles of a triangle?

17. Positive and Negative Segments.

1. If P_1, P_2, P_3, etc., are points equally spaced along a straight line, show that, when we take account of sign,

$$P_1P_5 + P_2P_1 = P_1P_4, \quad P_1P_3 + P_6P_1 + P_4P_7 = 0, \quad P_1P_4 + P_4P_2 = P_1P_2.$$

2. Show that if A, B, C, are any three points on a straight line, $AB + BC = AC$. Draw six figures to illustrate the different possible orders of the points.

18. Functions of Angles of Any Magnitude.

1. Write all the functions of 150° and of 210°. Which of these functions are equal except for sign? Which are equal?

2. Write the functions of 120° and of 300°. Which of these functions are equal except for sign? Which are equal?

3. Given $\cos x = -\frac{1}{2}$, construct geometrically the angle x, and compute the values of the other functions.

4. Construct the angle x, and compute the values of the primary functions in the following cases:

(*a*) $\sin x = +\frac{1}{2}$	(*e*) $\sec x = -\frac{5}{4}$	(*i*) $\csc x = +3$
(*b*) $\tan x = -\sqrt{3}$	(*f*) $\tan x = +\frac{2}{3}$	(*j*) $\operatorname{ctn} x = -\sqrt{3}$
(*c*) $\cos x = -\frac{1}{3}$	(*g*) $\cos x = +\frac{4}{5}$	(*k*) $\tan x = -10$
(*d*) $\operatorname{ctn} x = +\frac{3}{4}$	(*h*) $\sin x = -\frac{5}{6}$	(*l*) $\cos x = +0.3$

5. Find, without the use of the tables, the values of the functions of the following angles:

(*a*) 300°	(*e*) 240°	(*i*) 420°
(*b*) − 30°	(*f*) 150°	(*j*) − 750°
(*c*) − 120°	(*g*) − 225°	(*k*) − 45°
(*d*) 390°	(*h*) 135°	(*l*) 210°

6. Find the values of the primary functions of x when

(*a*) $\cos x = \frac{4}{5}$, and x lies in the first quadrant
(*b*) $\sin x = \frac{5}{13}$, and x lies in the second quadrant
(*c*) $\sin x = -\frac{3}{5}$, and x lies in the third quadrant
(*d*) $\tan x = -3$, and x lies in the second quadrant
(*e*) $\cos x = \frac{1}{3}$, and x lies in the fourth quadrant
(*f*) $\operatorname{ctn} x = \frac{3}{5}$, and x lies in the third quadrant
(*g*) $\sec x = -\frac{3}{2}$, and x lies in the third quadrant
(*h*) $\csc x = -5$, and x lies in the fourth quadrant

19. Projections.

1. AB makes a positive angle of 37° with the horizontal; BC, 20°; CD, − 15°; DE, − 110°. Find the length of the horizontal projection of the broken line $ABCDE$ when

(*a*) $AB = 20$	(*b*) $AB = 100$	(*c*) $AB = 26$
$BC = 50$	$BC = 27$	$BC = 9$
$CD = 5$	$CD = 16$	$CD = 35$
$DE = 8$	$DE = 12$	$DE = 48$

2. Find the projections on the north and south line, of the following broken lines:

(*a*) From A to B, N. 17° E., 35 feet
B to C, N. 72° E., 10 feet
C to D, S. 25° W., 54 feet *Ans.* − 12.38 feet.

(*b*) From A to B, N. 35° W., 55 feet
B to C, N. 2° W., 8 feet
C to D, N. 84° E., 7 feet
D to E, S. 89° W., 67 feet

(*c*) From A to B, S. 12° E., 19 feet
B to C, S. 77° E., 32 feet
C to D, N. 19° W., 75 feet
D to E, S. 10° W., 100 feet

3. The positive direction of the line L_2 makes the angle x with the positive direction of the line L_1. Find the numerical values and algebraic signs of the projections of the segment AB of L_1 upon L_2, and also of the segment AC of L_2 upon L_1 when

(*a*) $AB = 12$	(*c*) $AB = -6$	(*e*) $AB = 173$
$AC = 5$	$AC = 2$	$AC = -324$
$x = 27°$	$x = 172°$	$x = 200°$
(*b*) $AB = 110$	(*d*) $AB = 84$	(*f*) $AB = -41$
$AC = 57$	$AC = 21$	$AC = 12$
$x = 125°$	$x = 235°$	$x = 315°$

4. OP makes an angle of 33° with the horizontal, and PQ makes a positive right angle with OP. (That is, if we turn the segment OP through a positive right angle, it takes the direction of PQ, not of QP.) Find the horizontal projection of OPQ when $OP = 12$ and $PQ = 7$.

5. The line NP makes an angle x with the line MN, and MP makes an angle y with the horizontal. Find the horizontal and vertical projections of MNP when

(*a*) $MN = 5$	(*b*) $MN = 17$	(*c*) $MN = 105$	(*d*) $MN = 6$
$NP = 2$	$NP = 12$	$NP = 13$	$NP = 8$
$x = 90°$	$x = 120°$	$x = 150°$	$x = 130°$
$y = 40°$	$y = 18°$	$y = 110°$	$y = 210°$

6. A surveyor, starting from the point A, measures N. 29° E., 650 feet to B; thence E. 20° N., 510 feet to C; thence S. 15° W., 1000 feet to D; thence to A. Find the direction and distance from D to A. *Ans.* W. 22.57° N., 580.4 feet.

(SUGGESTION. Find the projections on the north and south line and on the east and west line.)

7. A surveyor measures from A, S. 52° E., 410 feet; thence S. 65° 35′ W., 275 feet; thence N. 44° 18′ W., 384 feet; thence back to A. Find the length and direction of the last line.

8. Find the fourth side and calculate the area of a field whose first three sides run respectively N. 70° 15′ W., 705.0 feet; S. 34° 23′ W., 497.5 feet; S. 65° 20′ E., 1287 feet.

9. Find the last side and area of the field from the following lines: From A, N. 21° E., 327.4 feet to B; thence N. 74° E., 343.8 feet to C; thence S. 77° 35′ E., 400.0 feet to D; thence S. 24° 18′ W., 673.2 feet to E; thence S. 15° 27′ E., 542.7 feet to F; thence S. 63° 24′ W., 341.3 feet to G; thence N. 57° 20′ W., 1241 feet to H; thence to A.

20. Line Values of Functions.

Establish the following relations by means of the line values:

1. $\sin 150° + \sin 210° = 0$.

2. $\tan 60° + \tan 300° = 0$.

3. $\sin 30° + \sin 150° = 1$.

4. $\cos 40° + \cos 140° + \cos 220° = -\cos 320°$.

5. $\csc^2 120° - \operatorname{ctn}^2 120° = 1$.

6. $\csc^2 225° + \csc^2 135° = (\operatorname{ctn} 225° - \operatorname{ctn} 135°)^2$.

7. $(\tan 20° - \tan 290°)^2 = \sec^2 20° + \sec^2 290°$.

8. $\sin 140° > \sin 160°$.

9. The numerical value of sin 130° is greater than the numerical value of sin 200°.

10. The numerical value of tan A is greater than the numerical value of sin A, when A is an angle of any quadrant.

11. Show that for angles less than 10°, the length of the arc which the angle intercepts in a unit circle does not differ from the sine of the angle by more than one unit in the third significant figure.

12. Show that, for angles less than 8°, the arc in a unit circle and the tangent of the angle do not differ by more than one unit in the third significant figure.

13. Show that, for angles less than 1.1°, the logarithm of the arc and the logarithm of the sine of the angle differ by less than half a unit in the fourth decimal place; and that the difference between the logarithm of the arc and the logarithm of the tangent of the angle is not appreciably greater than this.

14. In a unit circle

$$\frac{\text{length of the arc}}{\pi} = \frac{\text{number of degrees in the angle}}{180}.$$

Hence prove that, for all angles, log (length of arc of $x°$) $= \bar{2}.2419 + \log x$.

15. From the results of Exercises 13, 14, prove that, for angles less than 1.1°,

$$\log \sin x° = \bar{2}.2419 + \log x,$$
$$\log \tan x° = \bar{2}.2419 + \log x,$$
$$\log \operatorname{ctn} x° = 1.7581 - \log x.$$

16. Prove that, for angles between 88.9° and 90°,

$$\log \cos x° = \bar{2}.2419 + \log (90 - x),$$
$$\log \operatorname{ctn} x° = \bar{2}.2419 + \log (90 - x),$$
$$\log \tan x° = 1.7581 - \log (90 - x).$$

21. Reduction to Positive Acute Angles.

1. Express as functions of a positive acute angle:

(*a*) sin 170°	(*f*) cos 235°	(*k*) tan 173° 12′
(*b*) tan (− 98°)	(*g*) tan 268°	(*l*) sin 101.5°
(*c*) cos 135°	(*h*) ctn 273°	(*m*) ctn (− 183° 12′)
(*d*) csc (− 110°)	(*i*) sin 300°	(*n*) cos 154° 38′
(*e*) sin 210°	(*j*) sec (− 117°)	(*o*) tan 91.11°

2. Find, by means of the tables, the primary functions of the following angles: (*a*) 137°, (*b*) 212°, (*c*) 58°, (*d*) 301°, (*e*) 175°, (*f*) 271°, (*g*) − 35°, (*h*) − 192°, (*i*) 97°.

3. Given tan 226° = + 1.0355; find, without using the tables, tan 134°, tan 314°.

4. Given sin 105° = 0.966; find, without using the tables, all the primary functions of 255°.

5. Find the values of the following expressions:

(*a*) cos 135° + tan 315° − 2 sin 225° + tan 150°

(*b*) tan 120° − cos 240° + cos 210°

(*c*) $\sqrt{3}$ tan 150° − $\sqrt{2}$ sin 225° + $\sqrt{3}$ cos 210°

22. The Functions of $A \pm 90°$, $A \pm 180°$, etc.

1. What function of 10° is sin 100°?

2. What function of 20° is cos 160°?

3. What function of 22° is sec 202°?

4. Derive expressions for the functions of $A \pm 270°$ in terms of the functions of A.

5. Derive expressions for the following functions in terms of the functions of A:

(*a*) $\cos(A + 180°)$	(*h*) $\tan(180° - A)$	(*o*) $\sin(A - 90°)$
(*b*) $\tan(A + 180°)$	(*i*) $\sin(A - 180°)$	(*p*) $\cos(A - 90°)$
(*c*) $\sec(A + 180°)$	(*j*) $\csc(A - 180°)$	(*q*) $\tan(A - 90°)$
(*d*) $\operatorname{ctn}(A + 180°)$	(*k*) $\tan(A - 180°)$	(*r*) $\tan(A + 90°)$
(*e*) $\csc(A + 180°)$	(*l*) $\cos(A - 180°)$	(*s*) $\operatorname{ctn}(A + 90°)$
(*f*) $\sin(180° - A)$	(*m*) $\sec(A - 180°)$	(*t*) $\sec(A + 90°)$
(*g*) $\cos(180° - A)$	(*n*) $\operatorname{ctn}(A - 180°)$	(*u*) $\csc(A + 90°)$

6. Show that the formulas

$\sin(90° - A) = \cos A$ $\quad \tan(90° - A) = \operatorname{ctn} A$ $\quad \sec(90° - A) = \csc A$

$\cos(90° - A) = \sin A$ $\quad \operatorname{ctn}(90° - A) = \tan A$ $\quad \csc(90° - A) = \sec A$

which were established in § 2, when A is acute, hold for all values of A.

23. The Fundamental Identities. Applications.

No tables should be used in the following problems.

1. Find the other functions of x from the following data:

(*a*) $\sin x = +\frac{3}{5}$, x in the second quadrant

(*b*) $\tan x = -\sqrt{3}$, x in the fourth quadrant

(*c*) $\cos x = -\frac{1}{2}$, x in the third quadrant

(*d*) $\sec x = 2$, x in the fourth quadrant

2. Find all the values which the other functions of x may have in the following cases:

(*a*) $\tan x = -2$	(*c*) $\cos x = +\frac{2}{3}$	(*e*) $\operatorname{ctn} x = -\frac{1}{3}$
(*b*) $\sin x = -\frac{2}{3}$	(*d*) $\csc x = +\frac{5}{3}$	(*f*) $\sec x = +3$

3. Which sign should be used in the expression $\cos A = \pm\sqrt{1 - \sin^2 A}$ when A is an angle of the third quadrant?

4. Which sign should be used in the expression $\sec A = \pm\sqrt{1 + \tan^2 A}$ when A is in the second quadrant?

24. Circular Measure of Angles.

1. Express the following angles in circular measure as fractions of π: (*a*) 60°, (*b*) 30°, (*c*) 45°, (*d*) 75°, (*e*) 120°, (*f*) 150°, (*g*) 240°, (*h*) 270°, (*i*) 300°, (*j*) 145° 15′, (*k*) 22° 30′.

2. Express the following angles in degrees: (*a*) $\frac{1}{2}\pi$, (*b*) $\frac{1}{4}\pi$, (*c*) $\frac{3}{4}\pi$, (*d*) $\frac{1}{3}\pi$, (*e*) $\frac{2}{5}\pi$, (*f*) $\frac{1}{8}\pi$, (*g*) 2π, (*h*) 3π, (*i*) $\frac{5}{2}\pi$.

3. Express 1° in radians; 1′ in radians; 1″ in radians.

4. Express 1 radian in degrees and minutes; also in degrees correct to four significant figures.

25. Graphs of the Functions.

1. Plot the graph of sec x.
2. Plot the graph of ctn x.
3. Plot the cosecant curve.
4. Draw the graph of $\sin x + \cos x$.
5. Draw the graph of $\sin 2x$.
6. Draw the graph of $\tan 2x$.
7. Draw the graph of $2 \sin x - \sin 2x$.
8. Draw the graph of $\log x$.
9. Draw the graph of $\log_2 x$.
10. Draw the graph of 2^x.

26. Functions of the Sum and Difference of Two Angles.

1. Find sin 90°, cos 90°, tan 90° by using the fact that $90° = 30° + 60°$.

2. Find an exact expression in terms of radicals for cos 75° by using the fact that $75° = 45° + 30°$.

3. Find sin 75° in terms of functions of 45° and 30°.

4. Express sin 15° exactly in terms of radicals.

5. Express tan 15° exactly in terms of radicals.

6. Show that

(*a*) $\cos(x + 45°) = \dfrac{\cos x - \sin x}{\sqrt{2}}$ (*d*) $\tan(x + 45°) = \dfrac{1 + \tan x}{1 - \tan x}$

(*b*) $\sin(60° - x) = \dfrac{\sqrt{3}\cos x - \sin x}{2}$ (*e*) $\operatorname{ctn}(A + 30°) = \dfrac{\sqrt{3}\operatorname{ctn} A - 1}{\operatorname{ctn} A + \sqrt{3}}$

(*c*) $\tan(x - 60°) = \dfrac{\tan x - \sqrt{3}}{1 + \sqrt{3}\tan x}$ (*f*) $\sin(30° + A) = \dfrac{\cos A + \sqrt{3}\sin A}{2}$

7. Deduce the formulas of § 22 by means of the formulas for the functions of the sum and difference of two angles.

27. Functions of Multiples of an Angle.

No tables should be used in the following problems.

1. Given $\tan x = -2$, find the primary functions of $2x$, if x is an angle in the second quadrant.

2. Given $\sin x = \frac{1}{3}$, find $\cos 2x$.

3. Given $\tan x = 3$, find $\cos 2x$.

4. Given $\sin x = 0.3$, find $\sin \frac{1}{2}x$; $\cos \frac{1}{2}x$.

5. Find an expression for $\sin 90°$ and $\cos 90°$ by using the fact that $90° = 2(45°)$.

6. Find an expression for the primary functions of $180°$ by using the fact that $180° = 2(90°)$.

7. Find the value of the functions of $60°$ by using the fact that $60° = 2(30°)$.

8. Find the value of $\sin 120°$ and $\tan 120°$ by using the fact that $120° = 2(60°)$.

9. Show that $\cos 3x = 4\cos^3 x - 3\cos x$.

10. Find the value of $\sin 3x$ in terms of $\sin x$.

11. Show that $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

12. Find the value of $\cos 4x$ in terms of $\cos x$.

13. Find the value of $\sin 4x$ in the form of a product one of whose factors is $\cos x$, while the other factor is expressed in terms of $\sin x$.

14. Find the value of $\tan 4x$ in terms of $\tan x$.

Ans. $\dfrac{4\tan x(1 - \tan^2 x)}{1 - 6\tan^2 x + \tan^4 x}$.

15. Show that $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x$.

16. Find the values of $\cos 5A$ and $\tan 5A$.

17. Express all six of the trigonometric functions of x in terms of $\tan \frac{1}{2}x$.

18. Given $\sin 30° = \frac{1}{2}$; find $\sin 15°$, $\cos 15°$, $\tan 15°$, in radicals and hence in decimals.

Ans. $\sin 15° = 0.25882$,
$\cos 15° = 0.96592$,
$\tan 15° = 0.26795$.

19. Find the primary functions of $22.50°$ from the functions of $45°$, obtaining the answers first in terms of radicals and secondly in decimal form.

20. Prove that $\sin 7.50° = 0.1305$; $\cos 7.50° = 0.9914$.

21. From the results of Exercise 20, find $\sin 3.75°$; $\cos 3.75°$.

22. Prove that $\sin 18° = \frac{\sqrt{5}-1}{4}$. Hence compute the value of sin 18° in decimals to five significant figures.

(SUGGESTION. Let $x = 18°$. Then $2x = 36°$ and $3x = 54°$ are complementary angles. Consequently, $\sin 2x = \cos 3x$. Solve this equation for $\sin x$ by the aid of Exercise 9.)

23. Prove that $\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$. Hence compute the value of cos 18° in decimals to five significant figures.

24. From the results of Exercises 22 and 23 find in terms of radicals the primary functions of 36° and 54°. Reduce the results to their simplest forms.

25. From the results of Exercises 18, 22, 23, 24, compute a four-place table of sines and cosines at intervals of 3°. Check your results by comparison with a regular four-place table.

28. Sums and Differences of Sines and Cosines.

1. Express $2 \sin 53° \cos 17°$ as a sum of two sines, and hence compute its value.

2. Compute the values of the following quantities by expressing them as sums and differences of trigonometric functions:

(*a*) $\sin 47.31° \sin 23.15°$

(*b*) $\sin 76.20° \cos 43.10°$

(*c*) $\cos 33.52° \cos 21.33°$

3. Express $\sin x \cos y \cos z$ as a sum.

4. Express $\sin x \sin y \sin z$ as a sum.

5. Show that

(*a*) $\sin 45° + \sin 15° = \cos 15° = \frac{1}{4}\sqrt{2}(\sqrt{3}+1)$

(*b*) $\sin 45° - \sin 15° = \sqrt{3} \sin 15° = \frac{1}{4}\sqrt{6}(\sqrt{3}-1)$

(*c*) $\cos 45° + \cos 15° = \frac{1}{4}\sqrt{6}(\sqrt{3}+1)$

(*d*) $\cos 45° - \cos 15° = -\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$

6. Transform $\sin 50° + \sin 10°$ into a product.

7. Transform $\frac{\sin A + \sin B}{\cos A + \cos B}$ into a function of $\frac{1}{2}(A+B)$.

8. Prove that $\frac{\sin 22° + \sin 4°}{\cos 22° + \cos 4°} = \tan 13°$.

9. Prove that $\operatorname{ctn} 7° = \frac{\cos 21° + \cos 7°}{\sin 21° - \sin 7°}$.

10. Transform $\sin 5x + \sin 3x + \sin 2x$ into a product.

29. Trigonometric Identities.

Prove the following identities:

1. $\tan A \cdot \tan 2A + 1 = \sec 2A.$

2. $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A.$

3. $\dfrac{\sin 2A}{1 - \cos 2A} = \operatorname{ctn} A.$

4. $\dfrac{1 - \sin 2x}{1 + \sin 2x} = \left(\dfrac{\operatorname{ctn} x - 1}{\operatorname{ctn} x + 1}\right)^2.$

5. $2 \csc 2A = \tan A + \operatorname{ctn} A.$

6. $\sec 2x = \dfrac{1 + \tan^2 x}{2 - \sec^2 x}.$

7. $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}.$

8. $\pm\sqrt{\dfrac{1 + \sin A}{1 - \sin A}} = \dfrac{\cos A}{1 - \sin A}.$ When should the upper sign be used, and when the lower?

9. $\pm\sqrt{\dfrac{1 - \sin A}{1 + \sin A}} = \sec A - \tan A.$

10. $\operatorname{ctn}^2 x - \cos^2 x = \operatorname{ctn}^2 x \cos^2 x.$

11. $\dfrac{\cos x - \cos 3x}{\sin 3x - \sin x} = \tan 2x.$

12. $\dfrac{\cos 3x + \cos x}{\sin 3x - \sin x} = \operatorname{ctn} x.$

13. $\tan (x + 45°) + \tan (x - 45°) = 2 \tan 2x.$

14. If A, B, C are the angles of a triangle, prove that

(a) $\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$

(b) $\cos A + \cos B + \cos C = 1 + 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$

NOTE. If we were here to express the functions of $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$, in terms of the functions of A, B, C, radicals would be involved. These radicals will be avoided if we express the functions of A, B, C, in terms of those of $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$, by means of the formulas

$$\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A, \quad \cos A = \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A,$$

etc. This method of working is often convenient when the sines and cosines of half angles occur.

15. Prove that the sum of the tangents of the angles of a triangle is equal to the product of these tangents.

30. Trigonometric Equations.

Solve for x the following equations, obtaining *all* solutions in each case:

1. $\tan x - \sqrt{3} = 0$.
2. $\sin x = \frac{1}{2}\sqrt{3}$.
3. $\cos x = \frac{1}{2}\sqrt{2}$.
4. $\tan x = \sqrt{3}$.
5. $\cos^2 x = \frac{1}{2}$.
6. $\csc^2 x = 2$.
7. $\tan^2 x = 3$.
8. $\sin^2 x = \frac{1}{4}$.
9. $\tan^2 x = \frac{1}{3}$.
10. $\cos^2 x = 1$.
11. $3 \sin^2 x - 5 \sin x = -2$.
12. $\tan^2 x + 3 \tan x - 1 = 0$.
13. $3 \tan^2 x - 4 \sin^2 x - 1 = 0$.
14. $\cos^2 x - \tan^2 x = 1$.
15. $\sin x + \csc x = 3$.
16. $\cos x - \sec x = 5$.
17. $9 \sin x + 7 \cos x = 2$.
18. $3 \cos x + 8 \sin x - 6 = 0$.
19. $\frac{1}{3}\cos x - \frac{1}{4}\sin x = \frac{1}{5}$.
20. $\tan x - 2 = \sec x$.
21. $\tan 3x = \sqrt{3}$.
22. $\sin 2x = 0$.
23. $\cos 3x = \sin 2x$.
24. $\sin x + \sin 2x + \sin 3x = 0$.
25. $2 \sin x \sin 3x = 1$.
26. $\cos 3x = \cos 2x$.

Find all positive values of x less than 360° in the following equations:

27. $3 \operatorname{ctn} x - 2 \csc x + 1 = 0$.
28. $2 \sin 2x + 3 \sin x = 0$.
29. $3 \sin^2 2x + 2 \cos 2x - 2 = 0$.
30. $\sec x - \operatorname{ctn} x = \tan x - \csc x$.
31. $\cos 2x + \cos x + 1 = 0$.
32. $2 \cos 2x - \cos^2 x = 0$.
33. $2 \cos x \cos 3x + 1 = 0$.
34. $\tan 3x = 3 \tan x$.
35. $\cos 3x + 2 \cos x = 0$.
36. $\operatorname{ctn} x \tan 2x = \sec 2x$.
37. $\dfrac{1 - \tan x}{1 + \tan x} = \cos 2x$.
38. $\sin 2x + \cos 2x + \sin x = 1$.
39. $\tan x + \tan 2x = \tan 3x$.
40. $\sin 4x = \cos 3x + \sin 2x$.

31. Anti-trigonometric Functions.

1. Find the values of the following functions:

(*a*) $\sin^{-1}\frac{1}{2}\sqrt{3}$ (*c*) $\tan^{-1}\sqrt{3}$ (*e*) $\sec^{-1} 2$
(*b*) $\cos^{-1}\frac{1}{2}\sqrt{2}$ (*d*) $\operatorname{ctn}^{-1} 1$ (*f*) $\sin^{-1}(-\frac{1}{2})$

2. Show that

(*a*) $\sin(\cos^{-1}\frac{1}{2}) = \pm\frac{1}{2}\sqrt{3}$ (*c*) $\tan(2\cos^{-1}\frac{1}{2}) = \pm\sqrt{3}$
(*b*) $\tan(\sin^{-1}\frac{3}{5}) = \pm\frac{3}{4}$ (*d*) $\cos(2\tan^{-1}\frac{1}{3}\sqrt{3}) = \frac{1}{2}$

3. Find the values of the following expressions:

(*a*) $\sin(\cos^{-1}\frac{1}{2}\sqrt{3})$ (*c*) $\sin(\cos^{-1}\frac{3}{4})$
(*b*) $\cos(\tan^{-1} 1)$ (*d*) $\sin(2\tan^{-1} 2)$

4. Prove $\tan(\tan^{-1}x + \tan^{-1}y) = \dfrac{x+y}{1-xy}$.

5. $\sin(\cos^{-1}x) = \pm\sqrt{1-x^2}$.

6. $\cos(2\cos^{-1}x) = 2x^2 - 1$.

7. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45° + n\,180°$.

8. $\tan(2\tan^{-1}a) = \dfrac{2a}{1-a^2}$.

9. $\cos(2\tan^{-1}a) = \dfrac{1-a^2}{1+a^2}$.

10. Show that $\sin^{-1}x + \cos^{-1}x = 90°$, provided the anti-sine is taken as a positive or negative acute angle, and the anti-cosine as a positive angle less than 180°.

11. Show that

$$\tan^{-1}6 + \tan^{-1}(-4) = \tan^{-1}\tfrac{2}{23},$$

provided all three anti-tangents are understood to be positive or negative acute angles.

12. Show that

$$\sin^{-1}\frac{\sqrt{21}}{7} + \sin^{-1}\frac{\sqrt{21}}{14} = 60°,$$

provided that the two anti-sines are understood to be positive acute angles.

13. Show that $\tan\operatorname{ctn}^{-1}x = \dfrac{1}{x}$. Can we infer from this that $\operatorname{ctn}^{-1}x$ and $\tan^{-1}\dfrac{1}{x}$ are equal?

14. Prove that $\tan^{-1}x + \operatorname{ctn}^{-1}x = 90°$, provided suitable values for the anti-tangent and anti-cotangent are taken.

CHAPTER IV

32. The Law of Sines.

1. Apply the Law of Sines to the cases where $A = 90°$; $A = 0°$; $A = 180°$; and reduce the resulting formulas to their simplest forms.

2. Show that, in any triangle,

$$c = a\cos B + b\cos A,$$
$$a = b\cos C + c\cos B,$$
$$b = c\cos A + a\cos C.$$

NOTE. Relations of this sort between the sides and angles of a triangle which are to be established for all triangles, may sometimes be readily

proved by use of a figure, but a separate discussion of several different cases is then often necessary. Such relations may always be established without reference to a figure by remembering that, according to the Law of Sines, the sides are proportional to the sines of the angles. Thus the relation to be established is equivalent to another in which only the angles enter, and this, by means of the relation $A + B + C = 180°$, may be reduced to an ordinary trigonometric identity.

For example, the formula $c = a \cos B + b \cos A$ will, by the Law of Sines, be established if we can show that $\sin C = \sin A \cos B + \sin B \cos A$. But this, since $\sin C = \sin (A + B)$, is precisely formula (3) of § 26.

3. Prove, by means of the Law of Sines, that the bisector of an angle of any triangle divides the opposite side into segments proportional to the adjacent sides.

4. If O is the center of the circle circumscribed about any triangle ABC, and D is the middle point of BC, show that BDO is a triangle of reference for the angle A, so that, if R is the radius of the circumscribed circle,

$$\sin A = \frac{\frac{1}{2}a}{R}.$$

Hence show that the common value of the three members of the equation (2), § 32, which expresses the Law of Sines, is $2R$.

33. The Law of Cosines.

1. Apply the Law of Cosines, in the form involving $\cos A$, to the cases where $A = 90°$; $A = 0°$; $A = 180°$.

2. Prove that, in any triangle,

$$\cos A = \frac{s(s-a) - (s-b)(s-c)}{bc},$$

where $s = \frac{1}{2}(a + b + c)$.

3. Prove that in any triangle

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}.$$

4. Deduce the Law of Cosines, without reference to a figure, from the Law of Sines and the relation $A + B + C = 180°$. (See the note after Exercise 2, § 32.)

34. The Law of Tangents.

1. Apply the Law of Tangents in the form

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$$

in the cases where $C = 90°$; $A - B = 90°$; and reduce the results to their simplest forms.

2. Prove the identities

$$\frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} = \frac{\sin A - \sin B}{\sin (A+B)},$$

$$\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} = \frac{\sin A + \sin B}{\sin (A+B)}.$$

3. Hence show that in every triangle the relations hold

$$a - b = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} c,$$

$$a + b = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} c.$$

These formulas are of some importance, and may be called Napier's Analogies for plane triangles. See Chapter VI, § 54.

4. Deduce a proof of the Law of Tangents from the formulas of Exercise 3.

35. Graphical Solution of Triangles.

Solve graphically some of the triangles of the Exercises to §§ 36–39.

36. Case I. Two Angles and One Side.

Solve the following oblique triangles:

No.	Given			Required		
1.	$A=37.00°$	$B=75.00°$	$a=12.00$	$C=68.00°$	$b=19.26$	$c=18.49$
2.	$A=24.33°$	$C=63.41°$	$c=17.00$	$B=92.26°$	$a=7.832$	$b=18.99$
3.	$A=37.94°$	$C=41.32°$	$a=14.87$	$B=$	$b=$	$c=$
4.	$B=5.44°$	$C=97.21°$	$b=5431$	$A=77.35°$	$a=55900$	$c=56850$
5.	$B=107.32°$	$C=22.18°$	$c=5502$	$A=50.50°$	$a=11250$	$b=13920$
6.	$C=54.19°$	$A=100.02°$	$a=3.916$	$B=$	$b=$	$c=$
7.	$C=27.38°$	$B=95.49°$	$a=5113$	$A=$	$b=$	$c=$
8.	$B=15.27°$	$C=71.20°$	$a=45.01$	$A=93.53°$	$b=11.88$	$c=42.69$
9.	$A=13°\ 49'$	$C=80°\ 52'$	$c=118.2$	$B=85°19'$	$a=28.59$	$b=119.3$
10.	$B=91°\ 18'$	$A=35°\ 41'$	$c=57.29$	$C=$	$a=$	$b=$
11.	$B=15.81°$	$A=22.19°$	$a=5028$	$C=142.00°$	$b=3628$	$c=8196$
12.	$C=46.32°$	$B=49.18°$	$c=90.39$	$A=$	$a=$	$b=$
13.	$A=55°\ 23'$	$B=98°\ 32'$	$c=159.6$	$C=$	$a=$	$b=$
14.	$A=59.84°$	$C=102.88°$	$b=291.7$	$B=17.28°$	$a=849.4$	$c=957.4$
15.	$B=132°\ 12'$	$C=10°\ 41'$	$a=1.836$	$A=37°\ 7'$	$b=2.254$	$c=0.5638$
16.	$B=3.27°$	$A=5.83°$	$a=271.8$	$C=170.90$	$b=152.6$	$c=423.3$
17.	$C=54.32°$	$B=60.13°$	$a=5.401$	$A=$	$b=$	$c=$
18.	$B=64.18°$	$C=81.39°$	$a=1.043$	$A=34.43°$	$b=1.661$	$c=1.824$
19.	$A=165.31°$	$B=5.85°$	$a=8412$	$C=$	$b=$	$c=$
20.	$A=150.18°$	$C=2.18°$	$b=93.81$	$B=$	$a=$	$c=$

37. Case II. Two Sides and the Angle Opposite One of Them.

Solve the following oblique triangles:

No.	Given	Required		
1.	$a=32.61$ $b=17.38$ $A=52.91°$	$B=25.16°$	$C=101.93°$	$c=39.98$
2.	$a=177.0$ $b=216.0$ $A=35.60°$	$B_1=45.27°$	$C_1=99.13°$	$c_1=300.3$
		$B_2=134.73°$	$C_2=9.67°$	$c_2=51.08$
3.	$a=412.9$ $c=8923$ $C=67.31°$	$A=$	$B=$	$b=$
4.	$b=50.29$ $c=73.81$ $B=46.37°$	$C=$	$A=$	$a=$
5.	$a=73.80$ $b=54.29$ $B=44.26°$	$A_1=71.60°$	$C_1=64.14°$	$c_1=69.98$
		$A_2=108.40°$	$C_2=27.34°$	$c_2=35.72$
6.	$b=19.00$ $c=18.00$ $C=15°\ 49'$	$B=$	$A=$	$a=$
7.	$a=34.19$ $b=22.78$ $B=30°\ 18'$			
8.	$a=928.7$ $b=921.5$ $A=120.57°$	$B=58.68°$	$C=0.75°$	$c=14.12$
9.	$a=8403$ $b=4852$ $A=21.52°$			
10.	$a=71.43$ $b=116.5$ $A=81.32°$			
11.	$a=5553$ $c=6567$ $C=77.38°$	$A=55.60°$	$B=47.02°$	$b=4922$
12.	$b=24.32$ $c=29.31$ $B=47°\ 39'$			
13.	$b=91.05$ $c=77.03$ $B=51.11°$	$C=41.20°$	$A=87.69°$	$a=116.8$
14.	$a=107.3$ $c=171.2$ $C=0.951°$	$A=0.596°$	$B=178.453°$	$b=279$
15.	$b=3668$ $c=1523$ $C=55.90°$			
16.	$a=21.47$ $b=28.49$ $B=1.043°$			
17.	$b=8.123$ $c=6.072$ $B=126.32°$			
18.	$c=104.1$ $b=557.9$ $B=32.32°$	$C=5.73°$	$A=141.95°$	$a=643.1$
19.	$a=6.953$ $b=3.452$ $B=1°\ 4'$			
20.	$a=319.5$ $c=481.3$ $A=41.57°$			

38. Case III. Two Sides and the Included Angle

Solve the following oblique triangles:

No.	Given	Required		
1.	$a=3.421$ $b=5.179$ $C=51.27°$	$c=4.045$	$A=41.30°$	$B=87.43°$
2.	$a=27.81$ $c=39.53$ $B=45.32°$	$b=28.09$	$A=44.71°$	$C=89.97°$
3.	$b=7932$ $c=8731$ $A=120.41°$	$a=14460$	$B=28.22°$	$C=31.37°$
4.	$a=4129$ $c=4983$ $B=151.22°$	$A=$	$C=$	$b=$
5.	$c=1003$ $a=5101$ $B=98.35°$	$C=$	$A=$	$b=$

No.	Given	Required		
6.	$b = 99.83$ $c = 105.9$ $A = 46.32°$	$B =$	$C =$	$a =$
7.	$a = 327.9$ $b = 554.3$ $C = 67.32°$	$A = 35.27°$	$B = 77.41°$	$c = 524.1$
8.	$a = 5.001$ $b = 3.002$ $C = 67.32°$	$A = 76.90°$	$B = 35.78°$	$c = 4.739$
9.	$b = 39.28$ $c = 47.47$ $A = 14.25°$	$B =$	$C =$	$a =$
10.	$b = 1441$ $a = 9183$ $C = 108.35°$	$A = 63.57°$	$B = 8.07°$	$c = 9745$
11.	$a = 16.00$ $b = 12.00$ $C = 61.23°$	$A = 72.95°$	$B = 45.82°$	$c = 14.67$
12.	$a = 5.00$ $b = 7.00$ $C = 30° 15'$	$A =$	$B =$	$c =$
13.	$b = 40.00$ $c = 60.00$ $A = 80.00°$	$B =$	$C =$	$a =$
14.	$a = 33.00$ $c = 55.00$ $B = 25.00°$	$A = 29.07°$	$C = 125.93°$	$b = 28.71$
15.	$b = 25.00$ $c = 101.0$ $A = 35° 12'$	$B =$	$C =$	$a =$
16.	$a = 100.0$ $b = 300.0$ $C = 50.00°$	$A =$	$B =$	$c =$
17.	$b = 17.00$ $a = 19.00$ $C = 22.00°$	$A = 94.9°$	$B = 63.1°$	$c = 7.14$
18.	$b = 15.00$ $c = 21.00$ $A = 47.32°$	$B =$	$C =$	$a =$
19.	$b = 404.0$ $c = 505.0$ $A = 12.32°$	$B = 38.01°$	$C = 129.67°$	$a = 140.0$
20.	$a = 5112$ $c = 4343$ $B = 21.00°$	$A =$	$C =$	$b =$

39. Case IV. Three Sides.

Solve the following oblique triangles :

No.	Given			Required		
1.	$a = 12.00$	$b = 7.000$	$c = 9.000$	$A = 96.38°$	$B = 35.44°$	$C = 48.19°$
2.	$a = 41.00$	$b = 25.00$	$c = 18.00$	$A = 144.43°$	$B = 20.77°$	$C = 14.80°$
3.	$a = 33.00$	$b = 44.00$	$c = 55.00$	$A =$	$B =$	$C =$
4.	$a = 401$	$b = 303$	$c = 606$	$A =$	$B =$	$C =$
5.	$a = 2739$	$b = 4132$	$c = 1897$	$A = 32.84°$	$B = 125.10°$	$C = 22.06°$
6.	$a = 1221$	$b = 983.2$	$c = 4731$	$A =$	$B =$	$C =$
7.	$a = 9.478$	$b = 8.315$	$c = 3.241$	$A = 100.88°$	$B = 59.48°$	$C = 19.62°$
8.	$a = 4201$	$b = 3304$	$c = 5073$	$A =$	$B =$	$C =$
9.	$a = 41.00$	$b = 51.00$	$c = 101.0$	$A =$	$B =$	$C =$
10.	$a = 503$	$b = 703$	$c = 103$	Impossible.		
11.	$a = 3001$	$b = 4001$	$c = 6001$	$A =$	$B =$	$C =$
12.	$a = 5937$	$b = 9829$	$c = 7032$	$A =$	$B =$	$C =$
13.	$a = 15.45$	$b = 93.81$	$c = 85.48$	$A = 8.33°$	$B = 118.36°$	$C = 53.30°$
14.	$a = 5.728$	$b = 1.414$	$c = 1.732$	$A =$	$B =$	$C =$
15.	$a = 1728$	$b = 5280$	$c = 4629$	$A =$	$B =$	$C =$
16.	$a = 0.3125$	$b = 0.09831$	$c = 0.2738$	$A = 104.02°$	$B = 17.78°$	$C = 58.24°$

No.	Given			Required		
17.	$a=5.134$	$b=7.268$	$c=9.313$	$A=$	$B=$	$C=$
18.	$a=0.0135$	$b=0.0285$	$c=0.0196$	$A=24.8°$	$B=117.7°$	$C=37.5°$
19.	$a=193.5$	$b=335.2$	$c=225.1$	$A=$	$B=$	$C=$
20.	$a=9003$	$b=6051$	$c=12247$	$A=44.58°$	$B=28.14°$	$C=107.26°$

21. Prove the formulas

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}; \quad \cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}.$$

22. Prove that in a right triangle ($C = 90°$)

$$\tan \tfrac{1}{2} A = \sqrt{\frac{c-b}{c+b}}.$$

23. In a right triangle, given

$$b = 316.98, \qquad c = 317.26.$$

Compute A, first by the formula

$$\cos A = \frac{b}{c},$$

and secondly by the formula of Exercise 22. Show that the latter method gives about the same accuracy when we use three-place tables as the former does when we use five-place tables.

Notice that five-place tables are justified in the first method, since b and c are known to five significant figures, while there would be no object in using more than three-place tables in the second method, since $b - c$ is known to only two significant figures.

Note. The numerical example just discussed illustrates the fact that in solving right triangles, given a side and the hypothenuse, it is desirable to use the formula of Exercise 22 if these two lengths are nearly equal; the reason being that the determination of the very small angle A from its cosine is a delicate matter since a small change in the cosine produces a relatively large change in the angle, as may be seen from the table.

24. Let r_a denote the radius of a circle which touches the side a of a triangle ABC and the other two sides produced (escribed circle). Similarly let r_b and r_c be the radii of the other two escribed circles, which touch the sides b and c respectively and the other sides produced.

Prove that $$\tan \tfrac{1}{2} A = \frac{r_a}{s} = \frac{s-b}{r_a} = \frac{s-c}{r_b}.$$

25. By means of the formulas of Exercise 24 and formula (5), § 39, find values for r_a, r_b, r_c in terms of the sides of the triangle in forms analogous to formula (4), § 39, for r.

40. Area of a Triangle.

Find the areas of the following triangles:

No.	Given			Area
1.	$a = 8.000$	$b = 11.00$	$C = 47.35°$	32.36
2.	$a = 12.00$	$c = 26.00$	$B = 102° 17'$	152.4
3.	$a = 1385$	$b = 2791$	$C = 34° 23'$	
4.	$a = 59.12$	$c = 768.3$	$B = 46.31°$	
5.	$b = 5.428$	$c = 2.751$	$A = 163.15°$	2.164
6.	$b = 5.003$	$c = 7.014$	$A = 13.54°$	
7.	$a = 0.03210$	$b = 0.09680$	$C = 47.29°$	
8.	$a = 0.05436$	$c = 0.08431$	$B = 96.73°$	0.002276
9.	$b = 56.78$	$c = 86.75$	$A = 54° 39'$	
10.	$b = 5009$	$c = 9005$	$A = 72° 47'$	
11.	$a = 63.98$	$b = 47.25$	$c = 25.41$	517.8
12.	$a = 9.317$	$b = 8.429$	$c = 6.382$	
13.	$a = 0.0351$	$b = 0.3926$	$c = 0.3829$	0.006533
14.	$a = 725.9$	$b = 438.6$	$c = 1063.7$	
15.	$a = 0.9186$	$b = 0.7853$	$c = 1.2385$	0.3602
16.	$A = 54.67°$	$C = 82.35°$	$b = 905.3$	
17.	$a = 73.80$	$b = 54.29$	$B = 44.26°$	
18.	$a = 32.61$	$b = 17.38$	$A = 52.91°$	

19. By using the three different forms of formula (2), § 40, obtained by interchanging letters, deduce a new proof of the Law of Sines, and find a meaning in terms of the sides of the triangle and its area for the common value of the three members of equation (2), § 32.

By combining this result with Exercise 4, § 32, prove the formula

$$\text{Area} = \frac{abc}{4R},$$

where R is the radius of the circumscribed circle.

20. Prove that the area of a parallelogram is equal to the product of two adjacent sides by the sine of the included angle.

21. Prove that the area of any quadrilateral is equal to half the product of the diagonals by the sine of the included angle.

22. Prove the formulas

$$\text{Area} = s^2 \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C,$$

$$\text{Area} = \tfrac{1}{2}R(a \cos A + b \cos B + c \cos C),$$

$$\text{Area} = \frac{c^2 \sin A \sin B}{2 \sin C},$$

$$\text{Area} = Rr(\sin A + \sin B + \sin C),$$

$$\text{Area} = \frac{abc}{s} \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

Miscellaneous Oblique Triangles.

1. From the station A, a certain mountain peak has an angle of elevation of 38.47°. From a second station B, which is 13,000 feet farther away from the mountain in the same straight line with A and on the same horizontal plane, the angle of elevation of the peak is 24.32°. Find the height of the peak above the plane of A and B.

Ans. 13,620 feet.

2. Two points, B and C, are on opposite sides of a bay. From A, a point at the head of the bay, B is 12,380 feet distant, and C is 15,790 feet distant. From B, the lines of sight to A and C make an angle of 48.36°. Find the distance BC. *Ans.* 21,020 feet.

3. The two ends of a lake bear N. 35° W. and N. 67° E. from a station A. The distances from A to the ends are respectively 2738 feet and 4372 feet. Find the length of the lake.

4. A triangular piece of land has 187.35 feet frontage on one street, 143.9 feet on a second, and 247.3 feet on a third. Find the area of the piece. *Ans.* 13,420 square feet.

5. The distances from a point C to two points A and B, at opposite ends of a lake, are respectively 42,930 feet and 73,480 feet. The angle subtended by the lake at the point C is 24.8°. Find the distance AB.

Ans. 38,920 feet.

6. The centers of three towns are 1.9 miles, 3.4 miles, and 4.9 miles apart. What angles will straight roads between the towns make at the centers?

7. Two straight railroad tracks meet at an angle of 88.9°. If two trains start together from the junction, and travel at the rates of 27 and 18 miles an hour, how far apart will they be after twenty minutes?

Ans. 10.72 miles.

8. Two ships start from the same point. One sails E. 27° N., and the other E. 32° 54′ S. They each have wireless apparatus with a range of 250 miles. After sailing 185 miles, the first ship finds itself just out of

range with the second. How far has the second ship sailed, and how does it bear from the first one?

9. Two ships each have wireless apparatus with a range of 200 miles. One is 157 miles N. 23.42° E., and the other is 167 miles N. 48.35° W. from a shore station. Can the two ships communicate directly with each other?

10. Two objects, A and B, are in the same plane with two points, C and D, which are 500 feet apart. From C, the line BD subtends an angle of 15.21°, and DA subtends 127.10°. From D, the line AC subtends an angle of 42.19°, and CB subtends 143.14°. Find AB.

Ans. 2241 feet.

11. An aëroplane is observed simultaneously from two stations, A and B, whose horizontal distance apart is 6236 feet. A is 128 feet and B is 71 feet above sea level. From A, the aëroplane bears N. 2.97° E., and B bears East. From B, the aëroplane bears N. 21.25° W. The angle of elevation of the aëroplane at A is 17.68°, and at B is 16.77°. Find the altitude of the aëroplane above sea level. Use the angle of elevation at A, and check by using the angle at B.

Ans. 4646 feet; 4645 feet.

12. An observer on a train sees an object in a direction which makes an angle of 41.37° with the straight track ahead. After traveling 25 seconds at the rate of 30 miles an hour, the observer sees the same object at an angle of 73.12° with the track ahead. How far from the second position is the object?

13. Three circular Indian villages, each with a diameter of 381 feet, are situated on a plain in such a manner that their distances apart from center to center are 7.35 miles, 4.43 miles, and 9.63 miles. How large a triangular piece of land must be purchased in order to make one reservation, including the three villages? *Ans.* 10,500 acres.

14. The diagonals of a parallelogram are 76 and 54, and make an angle of 36.21° with each other. Find the lengths of the sides of the parallelogram.

15. Two lights, whose respective ranges are 32 miles and 18 miles, come into view from a ship at the same time. The first bears S. 28° E., and the second S. 58° W. How far apart are the lights?

Ans. 36 miles.

16. From a station 715 feet above sea level and 7695 feet from the base of a mountain, the angle of elevation of the summit is observed to be 29.37°. From the base, which is 1426 feet above sea level, the

summit has an angle of elevation of 58.94°. Find the distance from the base to the summit, and the altitude of the summit above sea level.

17. Two sides of a parallelogram are respectively 37.00 and 84.00, and the angle between them is 124.18°. Find the lengths of the two diagonals.

18. A and B are two towns on the opposite sides of a lake. The distance AB is 18.54 miles, and B bears E. 15° N. from A. By building a road from A 24.80 miles in a direction S. 17.44° E., a point C is reached at the end of the lake from which a straight road can be built to B. What will be the distance by road from A to B?

Ans. 55.13 miles.

19. The two diagonals of a parallelogram make an angle of 22.17° with each other. One diagonal is 3325 and one side is 1732. Find the other diagonal.

20. Two streets meet at an angle of 76.42°. How much land is included in the triangular corner lot which has a frontage of 327.4 feet on one street, and 297.8 feet on the other? *Ans.* 47,370 square feet.

CHAPTER V

A nautical mile is one minute of arc on a great circle of the earth's surface.

41. Geometrical Introduction.

1. Construct on a spherical surface two great circles making an angle of 23° with each other.

2. Mark three points, A, B, C, on a spherical surface and construct, first, the spherical triangle ABC; secondly, its polar triangle. Measure the sides and angles of these triangles, and show that the sides of each are the supplements of the angles of the other.

3. Sydney is in latitude 33° 52′ S., longitude 151° 13′ E., and San Francisco in latitude 37° 48′ N., longitude 122° 28′ W. Construct these points graphically on a spherical surface, and find by measurement the distance in nautical miles along a great circle between them.

4. With A as a pole and polar distance 40°, construct a small circle on the surface of a sphere. Divide this circle into six equal parts, and draw arcs of great circles connecting each pair of consecutive points.

42. Graphical Solution of Right Spherical Triangles.

1. Solve graphically the spherical right triangle in which $a=50°$, $c=85°$.

2. Given $a = 25°$, $A = 75°$, construct and solve graphically the two possible spherical right triangles.

3. Given $c = 65°$, $A = 30°$, solve graphically the spherical right triangle.

4. The two oblique angles of a spherical right triangle are 40° and 120°. Solve the triangle graphically.

5. Two points, A and B, on the earth's equator have longitudes 0° and 75° W., respectively. Find graphically the distance along a great circle from B to Greenwich, latitude 51° N.

43. The Trigonometric Formulas.

Establish the following formulas for spherical right triangles :

1. $\tan A \tan b \cos c = \sin a$.

2. $\sin c \cos A = \sin b \cos a$.

3. $\cos 2a - \cos 2c = 2 \cos^2 A \sin^2 c$.

44. Spherical Right Triangles which are Almost Plane.

1. A ship starting at a point on the equator sails due west a distance of 600 miles and then due north 800 miles. Find the distance between the point reached and the starting point, and determine the error which would be committed if the portion of the earth involved were regarded as plane.

2. Solve the above problem if the distances in question were 500 miles and 1200 miles.

45. The Trigonometric Solution of Spherical Right Triangles.

Solve the following spherical right triangles ($C = 90°$) :

No.	Given	Required
1.	$a = 117.52°$ $b = 63.18°$	$A = 114.94°$ $B = 65.85°$ $c = 102.03°$
2.	$a = 37.21°$ $b = 45.26°$	
3.	$b = 114.81°$ $c = 94.87°$	
4.	$b = 27.19°$ $c = 94.87°$	
5.	$a = 15.23°$ $c = 54.91°$	$A = 18.72°$ $B = 78.97°$ $b = 53.42°$
6.	$a = 133.13°$ $c = 73.12°$	
7.	$a = 47.33°$ $c = 83.91°$	
8.	$b = 35.35°$ $A = 87.87°$	$B = 35.40°$ $a = 86.32°$ $c = 87.00°$

No.	Given	Required
9.	$b = 114.30°$ $A = 63.49°$	
10.	$c = 48.57°$ $A = 105.28°$	$B = 157.57°$ $a = 133.69°$ $b = 163.37°$
11.	$c = 143.18°$ $A = 65.91°$	
12.	$c = 64.21°$ $A = 90.00°$	Impossible.
13.	$c = 90.00°$ $A = 90.00°$	
14.	$c = 45.34°$ $B = 17.83°$	$A = 77.26°$ $a = 43.93°$ $b = 12.58°$
15.	$c = 31.27°$ $B = 104.85°$	
16.	$a = 43.18°$ $A = 76.89°$	$B_1 = 18.13°$ $b_1 = 12.62°$ $c_1 = 44.64°$ $B_2 = 161.87°$ $b_2 = 167.38°$ $c_2 = 135.36°$
17.	$a = 171.12°$ $A = 126.14°$	
18.	$a = 35.18°$ $A = 35.18°$	
19.	$a = 115.35°$ $A = 145.33°$	
20.	$b = 114.03°$ $B = 114.03°$	$A = 90.00°$ $a = 90.00°$ $c = 90.00°$
21.	$A = 35.12°$ $B = 136.18°$	
22.	$A = 103.44°$ $B = 84.13°$	$a = 103.51°$ $b = 83.96°$ $c = 91.41°$
23.	$A = 144.33°$ $B = 31.19°$	

46. Spherical Oblique Triangle solved by Means of Right Triangles.

Solve the following quadrantal triangles:

No.	Given	Required
1.	$A = 51.18°$ $B = 163.49°$ $c = 90°$	$C = 53.06°$ $a = 77.12°$ $b = 159.17°$
2.	$a = 35.14°$ $B = 46.54°$ $c = 90°$	
3.	$B = 116.25°$ $C = 24.16°$ $c = 90°$	
4.	$A = 76.19°$ $a = 63.41°$ $c = 90°$	Impossible.
5.	$a = 37.65°$ $b = 41.22°$ $c = 90°$	
6.	$b = 71.33°$ $a = 132.41°$ $c = 90°$	$A = 135.40°$ $B = 64.30°$ $C = 72.03°$
7.	$B = 33.19°$ $A = 142.37°$ $c = 90°$	
8.	$b = 30.42°$ $A = 65.26°$ $c = 90°$	
9.	$A = 105.36°$ $a = 101.16°$ $c = 90°$	$B_1 = 134\ 09°$ $C_1 = 100.60°$ $b_1 = 133.06°$ $B_2 = 45.91°$ $C_2 = 79.40°$ $b_2 = 46.94°$
10.	$B = 43.61°$ $b = 69.32°$ $c = 90°$	
11.	$a = 28.76°$ $C = 45.24°$ $c = 90°$	

12. A quadrantal triangle has a vertex, A, at the north pole of the earth and another, B, on the equator. Let D be the point where the side AC (or AC produced) cuts the equator. Show that by solving the right triangle BDC we obtain easily the solution of the given triangle ABC.

Hence deduce a general method of reducing the solution of a quadrantal triangle to the solution of a right triangle without the use of the polar triangle.

Solve the following isosceles spherical triangles in which $b = c$:

13. $b = 54.33°$, $a = 72.90°$. *Ans.* $B = 57.99$, $A = 93.99°$.

14. $b = 57.21°$, $A = 132.67°$.

15. $b = 26.43°$, $B = 36.85°$.

16. $a = 147.28°$, $A = 182.95°$.

17. $a = 56.43°$, $B = 67.50°$. *Ans.* $b = 54.50°$, $A = 71.00°$.

18. $A = 55.32°$, $B = 113.19°$.

19. Five points, P_1, P_2, P_3, P_4, P_5, divide into equal parts a small circle whose polar distance is 23°. Find the distance in degrees from P_1 to P_2 measured along a *great* circle.

20. Find the ratio of the lengths (measured in some unit of length, not in degrees) of the arc P_1P_2 of the great circle to that of the arc P_1P_2 of the small circle of Exercise 19.

In problems like the following, a spherical triangle should be formed having the places mentioned as two vertices, and either the north or the south pole of the earth as the third.

21. Cape Henlopen, at the entrance of Delaware Bay, and Lisbon are both in latitude 38° 45′ N. The former is in longitude 75° 5′ W. and the latter in longitude 9° 11′ W. Find the distance in nautical miles between these places. How much shorter will the distance be which a vessel has to sail in going from Philadelphia to Lisbon if she sails on a great circle than if she steers a course due east after passing Cape Henlopen?

22. A vessel crosses the equator in longitude 41° 20′ W. How far has she to sail to reach Cape Town in latitude 33° 56′ S., longitude 18° 26′ E.?

Solve the following spherical oblique triangles by regarding them as the sum or difference of two right spherical triangles:

No.	Given			Required	
23.	$A=31.23°$	$C=122.12°$	$b=40.35°$	$a=33.15°$	$c=63.26°$ $B=37.88°$
24.	$A=50.17°$	$B=130.42°$	$c=60.25°$		
25.	$a=93.34°$	$b=56.56°$	$C=74.18°$		
26.	$a=40.27°$	$b=47.73°$	$C=79.88°$		
27.	$b=108.10°$	$c=40.32°$	$C=33.12°$		
28.	$a=16.10°$	$c=52.10°$	$A=15.63°$	$b_1=40.53°$	$B_1=39.15°$ $C_1=129.95°$
				$b_2=61.55°$	$B_2=121.32°$ $C_2=50.05°$
29.	$A=135.10°$	$B=146.22°$	$b=125.41°$		
30.	$b=48.33°$	$A=76.88°$	$B=59.80°$	$A_1=57.31°$	$C_1=66.53°$ $c_1=52.45°$
				$A_2=122.68°$	$C_2=152.24°$ $c_2=156.26°$

CHAPTER VI

47. The Law of Sines.

1. Show that sine formulas for right spherical triangles ((2), (3), § 43) are merely special cases of the Law of Sines.

2. By applying the Law of Sines to a quadrantal triangle, obtain expressions for the sines of its sides in terms of its angles.

3. Prove that in any spherical triangle the side which differs least from a quadrant is opposite the angle which differs least from a right angle, and the side which differs most from a quadrant is opposite the angle which differs most from a right angle.

4. Show that the Law of Sines for plane triangles is the limiting form of the Law of Sines for spherical triangles as the radius of the sphere becomes infinite.

48. The Law of Cosines.

1. Prove that

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A.$$

2. Prove that in any spherical triangle each angle is in the same quadrant as the opposite side, except that the angle which differs least from a right angle *may* be in a different quadrant from the opposite side.

3. Deduce the Law of Sines from the Law of Cosines without reference to a figure.

(SUGGESTION. Write the Law of Cosines first in the form involving A and then in the form involving B; and eliminate c between these two equations.)

4. Deduce the formulas for right spherical triangles (formulas (1)–(10), § 43) from the Law of Cosines.

On account of the fact that all the ordinary formulas of spherical trigonometry may be deduced from the Law of Cosines without the help of a figure (see Exercises 3, 4), this law is sometimes called the Fundamental Theorem of spherical trigonometry. Its fundamental importance makes a direct proof, such as is indicated in the next two exercises, of interest.

5. Assuming that b and c are both less than 90°, prove formula (1), § 48 by drawing the tangents at A to the arcs b and c, and extending the radii OB, OC until they meet these tangents in B' and C', respectively. Then apply the Law of Cosines for plane triangles to the two triangles $AB'C'$ and $OB'C'$.

6. Extend the Law of Cosines, as proved in Exercise 5, to the case in which b and c are both greater than 90° by extending the arcs AB and AC until they meet again in A', and applying the results of Exercise 5 to the spherical triangle $A'BC$.

Treat the case in which one of the two sides, b, c, is greater than 90°, the other less, by extending another pair of sides of the triangle.

49. Formulas for the Half-angles.

1. Establish the formulas

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}},$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}.$$

2. Deduce the half-angle formulas for plane triangles as limiting cases of the formulas of this section.

3. If a small circle is inscribed in the spherical triangle ABC, and r is the polar distance of this small circle, show that

$$p = \tan r.$$

4. Prove that in a right spherical triangle ($C = 90°$)

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (c - b)}{\sin (c + b)}},$$

and show how this formula may be used to advantage when b and c are nearly equal. Compare Exercises 22, 23, § 39.

50. The Solution of Spherical Triangles.

Compute the remaining parts of the following spherical triangles: *

No.	Given	Required
1.	$a = 80.30°$ $b = 61.33°$ $c = 114.45°$	$A = 62.64°$ $B = 52.24°$ $C = 124.90°$
2.	$a = 70.33°$ $b = 120.28°$ $c = 69.33°$	
3.	$a = 38.00°$ $b = 51.00°$ $c = 42.00°$	
4.	$a = 63.25°$ $c = 74.33°$ $B = 45.83°$	$A = 67.50°$ $C = 95.02°$ $b = 43.90°$
5.	$a = 40.27°$ $b = 47.73°$ $C = 79.88°$	
6.	$b = 60.33°$ $c = 112.42°$ $A = 64.77°$	
7.	$a = 117.15°$ $b = 27.37°$ $B = 22.33°$	$A = 47.35°$ $C = 146.67°$ $c = 138.33°$
8.	$b = 108.50°$ $c = 40.83°$ $C = 39.83°$	
9.	$a = 100.17°$ $c = 65.33°$ $C = 94.50°$	
10.	$A = 59.73°$ $B = 98.50°$ $C = 67.33°$	$a = 60.67°$ $b = 86.67°$ $c = 68.67°$
11.	$A = 87.00°$ $B = 110.00°$ $C = 60.85°$	
12.	$A = 51.97°$ $B = 83.90°$ $C = 58.88°$	
13.	$A = 101.40°$ $C = 74.67°$ $b = 56.50°$	$a = 93.33°$ $c = 79.17°$ $B = 54.97°$
14.	$B = 128.20°$ $C = 54.15°$ $a = 72.85°$	
15.	$A = 50.17°$ $B = 130.42°$ $c = 60.25°$	
16.	$A = 135.10°$ $B = 146.22°$ $b = 125.41°$	
17.	$A = 37.18°$ $C = 45.42°$ $a = 41.18°$	
18.	$B = 18.33°$ $C = 138.10°$ $c = 132.54°$	

19. Find the distance and direction from New York, latitude 40° 43′ N., longitude 74° W., to Seattle, latitude 47° 36′ N., longitude 122° 28′ W.

20. Find the distance and bearing of Gibraltar, latitude 36° 6′ N., longitude 5° 21′ W., from Boston, latitude 42° 21′ N., longitude 71° 41′ W.

21. Find the distance from Rio de Janeiro, latitude 22° 54′ S., longitude 43° 10′ W., to Lisbon, latitude 38° 43′ N., longitude 9° 11′ W.

* Exercises 23–30, § 46, may be used as additional exercises here.

22. Find the distance from San Francisco, latitude 37° 48′ N., longitude 122° 28′ W., to Manila, latitude 14° 35′ N., longitude 120° 58′ E.

51. Ambiguous Cases.

1. Give criteria according to which we can decide in Case IV whether a given triangle has 0, 1, or 2 solutions.

2. In those cases under Case IV where one of the solutions given by the trigonometric work should be discarded, give a criterion for deciding which solution should be retained.

52. The Law of Tangents.

1. Prove that in a right spherical triangle ($C = 90°$)

$$\tan(45° - \tfrac{1}{2}A) = \pm\sqrt{\frac{\tan\tfrac{1}{2}(c-a)}{\tan\tfrac{1}{2}(c+a)}},$$

and show how this formula may be advantageously used in certain cases in the solution of right spherical triangles.

2. Deduce the Law of Tangents for plane triangles as a limiting case of the formula of this section.

53. Dual Formulas.

1. Prove that in any spherical triangle

$$A + B - C < 180°,$$
$$A - B + C < 180°,$$
$$-A + B + C < 180°;$$

and show that if three angles, A, B, C, satisfy these inequalities, and also the inequality $A + B + C > 180°$ (see footnote, § 42), they are the angles of a spherical triangle.

2. Prove that in any spherical triangle the value of

$$\sin b \sin c + \cos b \cos c \cos A$$

is the same as the value of the similar expression for the polar triangle.

3. Prove that in a right spherical triangle ($C = 90°$)

$$\tan\tfrac{1}{2}b = \sqrt{\tan[\tfrac{1}{2}(A+B) - 45°]\tan[45° - \tfrac{1}{2}(A-B)]},$$

$$\tan\tfrac{1}{2}b = \sqrt{\tan\tfrac{1}{2}(c+a)\tan\tfrac{1}{2}(c-a)},$$

$$\tan\tfrac{1}{2}c = \sqrt{\frac{-\cos(A+B)}{\cos(A-B)}};$$

and show how these formulas may be advantageously used in certain cases in the solution of right spherical triangles.

54. Napier's Analogies.

1. Prove that the sum of any two sides of a spherical triangle is greater than, equal to, or less than 180° according as the sum of the opposite angles is greater than, equal to, or less than 180°.

2. Deduce the formulas of Exercise 3, § 34, as limiting cases of two of Napier's Analogies.

55. Improved Solutions of Spherical Triangles.

1. Solve by the methods of this section Exercises 4–22 of § 50.

2. Check the results of Exercises 1–3 of § 50 by the use of Napier's Analogies.

Date Due

APR 3 1964		
NOV 4 1975		
[illegible] 1977		
NOV 1977		
NOV 1977		
DEC 1977		
JAN 1978		

bdf CAT. NO. 23 233 PRINTED IN U.S.A.

TA 151 B63
Trigonometry, with the theory and u
Loeb Design Library AOJ7699
3 2044 026 850 06

TA
151
B63
2132